Math 250A

Professor Kenneth A. Ribet

Second Midterm Exam November 16, 1992

All modules below are *left* modules.

1 (5 points) Let A be an entire principal ring (principal ideal domain). Explain carefully what it means for an element of A to be an *irreducible* element. Suppose that π is such an element in A. Show directly (without invoking unique factorization) that the ideal (π) is maximal.

2 (10 points) Give examples of each of the following, including justification as appropriate:

- (2 pts.) a. A torsion-free $\mathbb{Z}$-module that is not free.
- (2 pts.) b. A factorial entire ring (unique factorization domain) that is not principal.
- (2 pts.) c. A Dedekind ring A that is not factorial. [Don't worry about proving that A is a Dedekind ring: just give some example.]
- (2 pts.) d. A ring A such that all A-modules are projective.
- (2 pts.) e. A ring A such that all A-modules are injective.

3 (9 points) Let M be a module over the commutative ring A. Let $\mathrm{Ann}(M)$ be the annihilator of M, i.e., the ideal $\{\, a \in A \mid am = 0 \text{ for all } m \in M \,\}$. Suppose that $\mathfrak{p}$ is a prime ideal of A, and let $M_{\mathfrak{p}}$ be the localization $S^{-1}M$ where $S = A \setminus \mathfrak{p}$.

- (4 pts.) a. Assume that $M_{\mathfrak{p}}$ is non-zero. Prove that $\mathrm{Ann}(M) \subseteq \mathfrak{p}$.
- (3 pts.) b. Suppose that $\mathrm{Ann}(M) \subseteq \mathfrak{p}$ and that M is finitely generated. Prove that $M_{\mathfrak{p}}$ is non-zero.
- (2 pts.) c. Suppose that $\mathrm{Ann}(M) \subseteq \mathfrak{p}$ but that M is not necessarily finitely generated. Is it always true that $M_{\mathfrak{p}}$ is non-zero?

4 (3 points) Let A be a commutative entire ring (integral domain). Let $\mathcal{F}$ be a covariant functor from the category of A-modules to the category of sets. Explain precisely what is meant by the statement that $\mathcal{F}$ is representable.

5 (3 points) Let X be the abelian group $\{\zeta \in \mathbb{C}^* \mid \zeta^{p^n} = 1 \text{ for some } n \geq 1\,\}$. Calculate the ring $R = \mathrm{End}(X)$. [First guess what R must be, and then try to prove that your answer is correct.]

☞ Homework due Friday, November 20: page 253, problems 2–4