Math 113H

Professor K. A. Ribet

Final Exam | May 24, 1991 | 12:30–3:30PM

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(3) **1.** Use quadratic reciprocity to determine whether or not 17 is a square mod 31.

(2) **2.** Write $x^6 + 18x^5 - 4x^3 + 2x + 22$ as a product of irreducible polynomials over J_2.

(2) **3.** Write $x^6 + 18x^5 - 4x^3 + 2x + 22$ as a product of irreducible polynomials over $\mathbf{Q}$.

(2) **4.** Write 17 as a product of irreducible elements in the ring $\mathbf{Z}[i]$.

(6) **5.** Find the minimal polynomial for $\sqrt{2} + \sqrt{3}$ over $\mathbf{Q}$ and show that it is irreducible over $\mathbf{Q}$.

(4) **6.** Let I be a left ideal of a ring R which contains $rs - sr$ for all $r, s \in R$. Show that I is a 2-sided ideal.

(8) **7.** Let N be a normal subgroup of a group G. Suppose that the order of N is 5 and that G has odd order. Prove that N is contained in the center of G.

(7) **8.** Let G be a subgroup of the additive group of real numbers such that G has only finitely many elements in each closed interval $[a, b]$. Prove that G is cyclic.

(8) **9.** Let $N = 561 = 3 \cdot 11 \cdot 17$. Prove the congruence $a^N \equiv a \pmod{N}$ for all integers a. (Assume known the corresponding result where N is replaced by a prime number.)

(9) **10.** Let g be an element of a finite group G, and let $\sigma\colon G \to G$ denote the permutation $x \mapsto gx$ of G. Express the sign of σ in terms of the order of g and the order of G.

(15) **11.** Let $R = \mathbf{Z}[\sqrt{5}]$ be the subring of $\mathbf{R}$ consisting of numbers $a + b\sqrt{5}$, with $a, b \in \mathbf{Z}$. Define a "norm" $N\colon R \to \mathbf{Z}$ by $N(a + b\sqrt{5}) = a^2 - 5b^2$.

a. Prove that an element of R is a unit if and only if its norm is ± 1.
b. Show that the group of units of R is infinite.
c. Show that no element of R has norm ± 2.
d. Show that the formula $a + b\sqrt{5} \mapsto (a + b \bmod 2)$ defines a ring homomorphism $\varphi\colon R \to J_2$.
e. Prove that the kernel of φ is not a principal ideal of R.

(9) **12.** Let K be a finite field. Prove that the product of the non-zero elements in K is -1.

The numbers to the left of the questions are the point values. The maximum possible score is 75.