Math 110

PROFESSOR KENNETH A. RIBET

Last Midterm Examination

April 3, 2014

9:40–11:00 AM, 105 Stanley Hall

Your NAME:

Your GSI:

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in *complete sentences*. Your explanations are your only representative when your work is being graded.

Vector spaces are over $\mathbf{F}$, where $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$. They may be infinite-dimensional if there is no indication to the contrary. Eigenvectors are non-zero!

At the conclusion of the exam, hand your paper in to your GSI.

Problem	Your score	Possible points
1		6 points
2		6 points
3		6 points
4		6 points
5		6 points
Total:		30 points

As a member of the UC Berkeley community, I act with honesty, integrity, and respect for others.

1. Label each of the following assertions as TRUE or FALSE. Along with your answer provide a correct justification or counterexample.

a. If S and T are operators on a finite-dimensional $\mathbf{F}$-vector space V such that $ST = 0$, then $TS = 0$.

The statement is false, as I noted in discussing a recent homework problem:

> ... one can find operators S and T such that $ST = 0$ but TS is non-zero. To see this, we can replace V by any F-vector space of dimension equal to $\dim V$, since two finite-dimensional vector spaces of the same dimension are isomorphic. I'll take V to be the polynomial space $P_m(F)$ with $m \geq 1$. (This space has dimension $m + 1$; recall that $\dim V$ is at least 2.) Let S be the differentiation operator $f \mapsto f'$ and let T be the operator that maps f to the number $f(0)$, regarded as a constant polynomial. Then $ST = 0$ because the derivative of a constant is 0, but TS is non-zero because a derivative might have non-zero constant term. (For instance, the derivative of x is 1.)

In the auditorium, someone asked whether "operator" means "linear operator." I said that it does, and I wrote something on the board to this effect. Here's Axler (p. 100):

> A linear map from a vector space to itself is called an operator.

So I don't feel guilty of introducing any ambiguity.

b. If S and T are operators on a finite-dimensional $\mathbf{F}$-vector space V such that $ST = I$, then $TS = I$.

This is true: if $ST = I$, then T is injective (1-1) and S is surjective (onto). It follows that both S and T are invertible because V is finite-dimensional. We have $T = S^{-1}$, so $TS = S^{-1}S = I$.

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2. Label each of the following assertions as TRUE or FALSE. Along with your answer provide a correct justification or counterexample.

a. If T is an operator on an $\mathbf{R}$-vector space V such that $f(T) = 0$ for some real polynomial $f(x)$ of odd degree, then T has at least one eigenvalue on V.

The statement seemed appealing (to me at least), but it's unfortunately not true. In fact, it's spectacularly false. For example, take $V = \{0\}$, $T = 0$ and $f(x) = x$. Then $f(T) = T = 0$, but T has no eigenvalues since there are no non-zero vectors in V.

b. If v_1, v_2 and v_3 are eigenvectors of T such that $v_3 = v_1 + v_2$, then all three vectors have the same eigenvalue.

We can replace v_3 by its negative, so that the relation among the three vectors is the symmetrical relation $v_1 + v_2 + v_3 = 0$. If two of the eigenvectors have the same eigenvalue λ, then the third vector also pertains to the eigenvalue λ. For example, if $Tv_1 = \lambda v_1$ and $Tv_2 = \lambda v_2$, then some high school algebra will show you that $v_3 = -(v_1 + v_2)$ is an eigenvector with eigenvalue λ. (A bit more conceptually, we could say that the eigenvectors with eigenvalue λ are the non-zero elements of the null space of $T - \lambda I$; this null space is a *subspace* and is, in particular, closed under addition and negation.)

The case to be excluded is that where the three vectors pertain to three different eigenvalues. However, we know from 5.10 on page 138 that eigenvectors whose eigenvalues are distinct cannot be linearly dependent. In particular, the sum of three of them cannot be 0.

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3. Let U be a subspace of an $\mathbf{F}$-vector space V. Assume that $(v_1 + U, \ldots, v_t + U)$ is a basis of V/U and that $(u_1, \ldots, u_s)$ is a basis of U. Prove that $(u_1, \ldots, u_s; v_1, \ldots, v_t)$ is a basis of V.

OK, let's plod through this:

First, for linear independence let's assume that we have $0 = a_1u_1 + \cdots + a_su_s + b_1v_1 + \cdots + b_tv_t$. Passing to V/U, we get $(b_1v_1 + \cdots + b_tv_t) + U = 0$, where the 0 on the right-hand side of the equation is the 0-element of V/U (namely, U). Because $(v_1 + U, \ldots, v_t + U)$ is linearly independent, all the b_i are 0. Then we have $0 = a_1u_1 + \cdots + a_su_s$, and we get that the a's are zero as well because of the linear independence of $(u_1, \ldots, u_s)$.

Second, we have to check that $(u_1, \ldots, u_s; v_1, \ldots, v_t)$ spans V. Take $v \in V$ and consider the image $v + U$ of v in V/U. We there are scalars b_i so that $v + U = (b_1v_1 + \cdots + b_tv_t) + U$ because $(v_1 + U, \ldots, v_t + U)$ spans V/U. This means that there is some $u \in U$ so that $v = u + b_1v_1 + \cdots + b_tv_t$. Because $(u_1, \ldots, u_s)$ spans U, we can write u as a linear combination of the u_i. Putting everything together, we have $v = a_1u_1 + \cdots + a_su_s + b_1v_1 + \cdots + b_tv_t$, where the a's and b's are scalars.

This problem was pretty routine—sorry!

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4. Let T be an operator on an $\mathbf{F}$-vector space V. Assume that $(T - aI)(T - bI) = 0$, where a and b are scalars in $\mathbf{F}$. If T is not a scalar multiple of the identity operator, prove that a and b are eigenvalues of T.

If T is not a scalar multiple of I, then T isn't aI; in more mathematical terms, $T - aI$ is a non-zero operator. This means that there is a $v \in V$ for which $(T - aI)v$ is non-zero. This vector is in the null space of $T - bI$ because of the hypothesis that $(T - aI)(T - bI)$ is 0. Hence b is an eigenvalue of T. Exchanging the roles of a and b, we see similarly that a is an eigenvalue of T. [Note that $(T - aI)(T - bI) = (T - bI)(T - aI)$.]

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5. Suppose that $V \subset \mathbf{F}^n$ is a proper subspace (i.e., a subspace not equal to all of $\mathbf{F}^n$). Show that there are scalars $c_1, \ldots, c_n$, not all of which are zero, so that $c_1a_1 + c_2a_2 + \cdots + c_na_n = 0$ for all vectors $(a_1, \ldots, a_n)$ in V.

It was somehow not apparent in the auditorium that the a_i were scalars and that $(a_1, \ldots, a_n)$ was intended to be an element of $\mathbf{F}^n$ that happens to be in V. I wrote a clarification on the board for people who were misled.

One way to do this problem is to say that each vector $c = (c_1, ..., c_n)$ defines a linear map $\mathbf{F}^n \to \mathbf{F}$ by the formula $(a_1, \ldots, a_n) \mapsto a_1c_1 + \cdots a_nc_n$. Restricting this map to V yields a linear form $\varphi_c : V \to V$. Intrinsically, we get in this way a linear map

$$T : \mathbf{F}^n \to V', \qquad c \mapsto \varphi_c.$$

What is wanted is that there is some non-zero c for which φ_c is the zero map. In other words, we have to see that $\text{null}(T) \neq \{0\}$. However, because V is a proper subspace of $\mathbf{F}^n$, $\dim V < n$. Thus $\dim V' = \dim V < n$, so that T is mapping an n-dimensional space to a lower-dimensional space. By the rank–nullity formula, the null space of T must be non-zero.

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