Math 110

PROFESSOR KENNETH A. RIBET

Last Examination

May 14, 2014

11:30AM–2:30PM, 230 Hearst Gym

Your NAME:

Your GSI:

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly in *complete sentences.* (There is lots of time!) Your explanations are your only representative when your work is being graded.

Vector spaces are over $\mathbf{F}$, where $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{C}$. They may be infinite-dimensional if there is no indication to the contrary. A "real" vector space is a vector space over $\mathbf{R}$.

At the conclusion of the exam, hand your paper in to your GSI.

Problem	Your score	Possible points
1		8 points
2		8 points
3		8 points
4		8 points
5		7 points
6		7 points
Total:		46 points

1. Label each of the following assertions as TRUE or FALSE. Along with your answer provide a proof or counterexample.

a. Suppose that $T \in \mathcal{L}(V)$ with V of finite dimension n. If $V = \operatorname{range}(T) \oplus \operatorname{null}(T)$, then $\operatorname{null} T = \operatorname{null} T^2 = \operatorname{null} T^3 = \cdots = \operatorname{null} T^n$.

> This is correct. Suppose that $T^2 v = 0$. Then $Tv \in \operatorname{range}(T) \cap \operatorname{null}(T) = \{0\}$, so $Tv = 0$. Hence any vector in $\operatorname{null} T^2$ is already in $\operatorname{null} T$.

b. If V is a finite-dimensional complex vector space, and if T is an operator on V, then T^k is diagonalizable for some positive integer k.

> No, this is not true. For example, let T be the operator on $\mathbf{C}^2$ with matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then T^k has matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and thus isn't diagonalizable.

2. Label each of the following assertions as TRUE or FALSE. Along with your answer provide a proof or counterexample.

a. If $V = U \oplus X$ and $V = U \oplus Y$, then $X = Y$.

> This is false, as was discussed on piazza on or around May 9. If $V = \mathbf{R}^2$ and U is the line generated by e_1 (i.e., the x-axis), then we can let X be any one-dimensional subspace of V other than U and we'll have $V = X \oplus U$.

b. If V is a finite-dimensional vector space and U is a subspace of V, then every linear functional on U can be extended to a linear functional on V.

> This is true, as I hope most of you recognized. Take a basis $(v_1, \ldots, v_d)$ of U and extend it to a basis $(v_1, \ldots, v_n)$ of V. This is kind of a knee-jerk thing to do, but once you've done it you're in good shape and can finish the proof in various ways. For example, the linear map $P : V \to U$, $a_1 v_1 + \cdots + a_n v_n \mapsto a_1 v_1 + \cdots a_d v_d$ is the identity on U; it's the projection of V onto U along $\operatorname{span}(v_{d+1}, \cdots, v_n)$. If $\varphi : U \to \mathbf{F}$ is a linear functional on U, $\varphi \circ P$ is an extension of φ to all of V.

3. Label each of the following assertions as TRUE or FALSE. Along with your answer provide a proof or counterexample.

a. If an operator on a real finite-dimensional inner product space V has a symmetric matrix with respect to one orthonormal basis of V, then it has a symmetric matrix with respect to all orthonormal bases of V.

> This is certainly true: If $(e_1, \ldots, e_n)$ is an orthonormal basis of V, the matrix of $T \in \mathbf{L}(V)$ with respect to this basis is symmetrix if and only if T is self-adjoint. The self-adjointness condition does not reference any particular basis; the symmetry property in one basis implies the symmetry property in any other.

b. A normal operator on a complex finite-dimensional inner-product space is self-adjoint if and only if all of its eigenvalues are real.

Choose an orthonormal basis of the space in which the operator is diagonal; this is possible by the complex spectral theorem. In this representation, the distinct diagonal entries are the distinct eigenvalues of the operator. Also, the operator's adjoint is represented in this basis by the adjoint of this diagonal matrix, which is just the matrix gotten by replacing the diagonal entries by their complex conjugates. Thus the diagonal entries are all real if and only if the matrix representation of the adjoint is the matrix representation of the operator. Said in other terms, the eigenvalues are all real if and only if the operator is equal to its adjoint.

4. Label each of the following assertions as TRUE or FALSE. Along with your answer provide a proof or counterexample.

a. If v is a non-zero vector in a finite-dimensional vector space V, there is a basis $(v_1, \ldots, v_n)$ of V such that

$$v = v_1 + v_2 + \cdots + v_n.$$

This is true, as explained in the solution to exericse 3.C.6 that I circulated on or around March 26. Use v as the first element of a basis $(v, v_2, v_3, \ldots, v_n)$ As an alternative basis, consider $(v - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$. The sum of the vectors in this basis is v. Also, some fooling around should convince you that the list I've just written down is linearly independent.

b. If U is a finite-dimensional subspace of an inner-product space V (possibly infinite-dimensional), then $V = U \oplus U^{\perp}$.

This is true; it's 6.46 on page 194 of the book. Just knowing this should get you most of the way toward full credit for the problem. To get all possible points, you should say something in the direction of why the statement is true.

5. Suppose that T is an operator on a finite-dimensional real vector space V and that T satisfies $T^2 + 4T + 5I = 0$.

a. If V is non-zero, show that there is a T-invariant subspace of V whose dimension is 2.

An important point is that T has no eigenvalues on V: if λ were an eigenvalue, we'd get $\lambda^2 + 4\lambda + 5 = 0$, which is impossible because $x^2 + 4x + 5$ has a negative discriminant. To do the proof that's required, we take a non-zero vector v in V and note that v isn't an eigenvector, which means that the list (v, Tv) is linearly independent. The span of (v, Tv) is therefore a two-dimensional subspace of V. It's T-invariant because $T(Tv) = T^2 v = -(4Tv + 5v)$ is in the span.

b. Show that $\dim V$ is even.

We do this by induction on $\dim V$. The base case is that where $\dim V = 0$; there is nothing to prove here because 0 is an even number. In the induction step, we assume that $\dim V$ is positive and that the parity condition has been established for all vector spaces of dimension less than $\dim V$ that come equipped with an operator satisfying the quadratic relation in the problem. Let U be a subspace of dimension 2 as in part (a) and consider the space V/U, whose dimension is $\dim V - 2$. The operator T/U satisfies the quadratic equation that is satisfied by T; hence $\dim V - 2$ is even, by the inductive hypothesis. Thus $\dim V$ is itself even.

6. Suppose that V and W are finite-dimensional vector spaces and that T_1 and T_2 are linear maps from V to W. If the range of T_1 is contained in the range of T_2, show that there is an operator $S \in \mathcal{L}(V)$ such that $T_1 = T_2 S$.

See the solution to exercise 3.D.5 that I distributed in mid-March.