Math 110
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Linear Algebra
Spring, 2005

January 20, 2005

A psychologist named Nalini Ambady gave students three 10-second soundless videotapes of a teacher lecturing. Then she asked the students to rate the teacher. Their ratings matched the ratings from students who had taken the teacher's course for an entire semester. Then she cut the videotape back to two seconds and showed it to a new group. The ratings still matched those of the students who'd sat through the entire term.

—Review of Malcolm Gladwell's "Blink: Hunch Power"

I should have mentioned in my lecture on Tuesday that Malcolm Gladwell is coming to Cody's on Telegraph on January 20, 2005. That's tonight! His presentation starts at 7:30PM.

These "slides" were prepared before class on Tuesday. They will represent some part of my lecture on Thursday, January 20. However, I intend to lecture using chalk boards on January 20.

Subspaces

An important concept in linear algebra is that of a *subspace* of a vector space V. (The field F is part of the picture, but goes unmentioned in the name of the concept.) A subspace of V is a non-empty subset $W \subseteq V$ that is closed under addition, subtraction and scalar multiplication. A subspace automatically contains 0, because $0 = x - x$ if x is in W.

Two obvious subspaces of V are V itself and the subset $\{0\}$. A *proper* subspace is one other than V.

Examples

If v is an element of V, then the set $\{\, av \mid a \in F \,\}$ is a subspace of V. This is 0 if $v = 0$ and is the "line generated by v" if v is non-zero.

For $n \geq 1$, the set of vectors in F^n whose nth coordinates are 0 is a subspace of F^n. It's pretty much indistinguishable from F^{n-1}.

In the same vein, you get lots of subspaces of $M_{n\times n}(F)$ or even $M_{m\times n}(F)$ by looking at the set of matrices where certain pre-determined entries are 0. For example, the upper-triangular matrices form a subspace of $M_{n\times n}(F)$. The diagonal matrices form a subspace of $M_{n\times n}(F)$. In fact, this latter subspace is also a subspace of the space of upper-triangular matrices. One could go on and on here.

A key point is that there's no need to make entries be 0. We can do something elaborate. For example, we could let

$$W = \{\, (x, y, z, w) \in F^4 \,|\, x + 2y + 3z + 4w = 0 \,\}$$

or we could consider the set of 2×2 matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

[What wouldn't work would be to take the set of (x, y, z, w) with $4x+3y+5z+9w = 2$, for example: the set wouldn't be closed under addition.]

Very generally, if $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ (a matrix with m rows and n columns), then the set of $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in F^n$ such that $Ax = 0$ is the subspace of F^n consisting of all $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ such that $\sum_j a_{ij}x_j = 0$ for each i, $1 \leq i \leq n$.

The map $x \mapsto Ax$ is a function $F^n \longrightarrow F^m$. You may know from Math 54 that this function is a linear map (or linear transformation) $F^n \longrightarrow F^m$ and that the subspace of F^n that we just defined (i.e., the set of x mapping to 0) is the kernel of this linear transformation.

Mantra: kernels are subspaces.

Subspaces from subspaces

If W_1 and W_2 are two subspaces of V, then so are $W_1 \cap W_2$ and $W_1 + w_2 := \{\, x + y \mid x \in W_1, y \in W_2 \,\}$. In fact, the intersection of an arbitrary collection of subspaces of V is again a subspace of V.

This sounds innocuous, but a consequence is that, for each subset S of V, there is a smallest subspace of V that contains S: it's the intersection of all of the subspaces that do contain S.

We want to understand this subspace explicitly. Call it W. If $s_1, \dots, s_t$ are elements of S and $a_1, \dots, a_t$ are numbers in F, then W contains the *linear combination* $a_1 s_1 + \cdots + a_t s_t$. (If $t = 0$, this linear combination is the empty sum, which is deemed to be 0.) Thus W contains all linear combinations of elements of S with coefficients in F.

Further, the set of all such linear combinations forms a subspace of V that contains S, so it must contain W (= the smallest subspace. . .).

Conclusion: W *is* the set of these linear combinations! The smallest subspace of V that contains S consists of the linear combinations of elements of S with coefficients in F. These are finite sums $\sum_i a_i s_i$ as on the previous slide.

The smallest subspace of V that contains S is called the *span* of S; it's written $\mathrm{span(S)}$ in our book. We say that $\mathrm{span(S)}$ is the subspace of V that is *generated by* S. If $\mathrm{span(S) = V}$, then V is generated by S and S is a *generating set* for V.

Linear Independence

There are a couple of allied concepts, and it's important to pay attention. First, suppose that we have a list of vectors in V: $v_1, v_2, \ldots, v_n$. We say that $v_1, v_2, \ldots, v_n$ are linearly dependent if there are $a_1, \ldots, a_n \in F$, not all 0, such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0.$$

Equivalently, $v_1, v_2, \ldots, v_n$ are linearly dependent if one of the v_i is a linear combination of the others. Indeed, if $a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0$ and a_i is non-zero, then we can divide by a_i and write

$$v_i = -(1/a_i) \sum_j a_j v_j,$$

where the sum is taken over the j different from i. Conversely, if $v_i = \sum_{j \neq i} c_j v_j$, then $0 = -1 \cdot v_i + \sum_{j \neq i} c_j v_j$.

We say that $v_1, \ldots, v_n$ are linearly independent if they are not linearly dependent. This means:

$$0 = \sum_i a_i v_i \Longrightarrow a_1 = a_2 = \cdots = a_n = 0.$$

It is equivalent to say that the map

$$(a_1, \ldots, a_n) \mapsto \sum_i a_i v_i, \quad F^n \to V$$

is injective (i.e., 1-1). The proof is easy but needs to be supplied.

Here is a silly special case: suppose that we have a list $v_1, \ldots, v_n$ and that two vectors on the list are equal. Say $v_i = v_j$ with $i \neq j$. Then we have $0 = 1 \cdot v_i + (-1) \cdot v_j$; this is a non-trivial linear combination that is 0, so the vectors $v_1, \ldots, v_n$ are linearly dependent.

Another observation: if one of the vectors is 0, then $v_1, \ldots, v_n$ are linearly dependent. Indeed, if $v_i = 0$, then $0 = 1 \cdot v_i$.

One rarely worries about linear independence of vectors when one of the vectors is 0.

Also, people usually take distinct vectors when they consider the linear independence of a bunch of vectors $v_1, \ldots, v_n$. In our text, I am told that the authors sometimes write down a list of vectors and assume tacitly that the vectors are distinct without mentioning explicitly that they are making this hypothesis. So watch out.

Suppose that $v_1, \ldots, v_n$ are elements of F^m; view them as n columns of length m. Stack them in a line to get an $m \times n$ matrix $C = v_1 \cdots v_n$. Think of the coefficients $a_1, \ldots, a_n$ as a column vector A. Then $a_1 v_1 + \cdots + a_n v_n = 0$ if and only if $C \cdot A = 0$. This means that $(a_1, \ldots, a_n)$ form a solution to a homogeneous system of linear equations in n unknowns $a_1, \ldots, a_n$. There are m equations in the system. Whether or not the system has a non-trivial (=non-zero) solution is the question of whether or not the v_i are linearly dependent.

Here is the place where we need to pay attention. If S is a subset of V, we say that S is linearly dependent if there are distinct vectors $v_1, \ldots, v_n$ in S and $a_1, \ldots, a_n$ in F, not all 0, such that $a_1v_1 + \ldots + a_nv_n = 0$. We say that S is linearly independent if it is not linearly dependent.

In this second definition, we are talking here about a *set* of vectors. In the previous definition, we were talking about a finite list of vectors. The set S here can be infinite. Even if it's finite, it doesn't come with an order of any kind. If you have a list of 15 vectors v_i, and they are all equal, then the set of the v_i has only one element.

For example, in $\mathbf{R}^2$, the two vectors $(0,1)$ and $(1,0)$ are linearly independent. The vectors $(0,1)$, $(0,1)$, $(1,0)$ are linearly dependent. The set $S = \{\, (0,1), (0,1), (1,0) \,\}$ is linearly independent. It has two elements!

A good example of an infinite linearly independent set occurs if you take V to be the space of polynomials (in x) over F. The set

$$S = \{\, 1, x, x^2, x^3, \dots \,\}$$

is linearly independent. The proof is enough of a tautology that it should be pondered: to say that a linear combination of a finite set of the x^i is 0 in V is to say that some polynomial $c_n x^n + \cdots + c_0$ is 0. However, in V, a polynomial is deemed to be 0 if and only if all of its coefficients are 0.

This is a side topic, but it's important. Suppose that $F = \{0, 1\}$ is the field with two elements. Let $f(x) = x^2 - x$. This is a non-zero polynomial. It has degree 2. We love this polynomial. However, it vanishes at all elements of F: we have $f(0) = f(1) = 0$.

Conclusion: it is possible for a polynomial to give the zero-function on F and yet not be the zero polynomial. This can happen only when F is finite, however.

In the definition of linear dependence of a set, the number n ($=$ the length of the list of the v_i) is not given in advance. If there is a non-trivial linear combination of 10^6 different vectors in S that vanishes, then S is linearly dependent. If S is finite, we can always take n to be the number of elements of S.