Math 110
Professor Ken Ribet

Linear Algebra
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See `http://math.berkeley.edu/~ribet/110/` for information about:

- The textbook
- The discussion sections
- Exam dates
- Grading policy
- Office hours (link)

Note especially that there is an online discussion group, Google's **Math 110**. The URL for the group is `http://groups-beta.google.com/group/Math110/`. If you "join" the group, you can post comments and questions. If you don't join, you can still lurk and read what other people have written.

We have a fine lineup of experienced GSIs: Chu-Wee Lim, Scott Morrison, John Voight. Discussion sections are on Wednesdays; see the class Web page for times and room numbers. Note that #103 is in 433 Latimer, not 435 Latimer.

Some of the sections are full. If you want to change your section to one that is full, you need to speak with Barbara Peavy in 967 Evans. The burden will be on you to convince her that your schedule requires you to change into one of the full sections.

An initial question for feedback: Is lecturing with a laptop going to be effective? Mathematicians have traditionally written on the board with railroad chalk (fat chalk) when giving large courses. Should I do that? Should I use transparencies? Some combination? Post to the newsgroup with your thoughts.

Meanwhile, I will try to lecture today with a laptop and the projector.

Even though we're in a large room, don't hesitate to stop me to ask questions. One good thing about the laptop is that I can see you while I'm speaking. You can interrupt—don't be shy. If you would like some clarification, so will your friends.

Vector Spaces and Fields

I imagine that you know more than a bit about matrices, systems of equations, linear maps,. . . . You have taken Math 54. In Math 110, we study vector spaces and linear transformations more abstractly. We will be interested a choosing bases for vector spaces in such a way that linear transformations are represented by especially nice matrices.

What is a field? It's a set with an addition and a multiplication; the system is required to satisfy a list of familiar-looking axioms (Appendix C of book). Some examples:

- The field $\mathbf{R}$ of real numbers.
- The field $\mathbf{C}$ of complex numbers.
- The field $\{0, 1\}$ with two elements.
- The field $\mathbf{Q}$ of rational numbers.

Fix a field F. A vector space over F is a set V together with two additional structures: an addition law $(x, y) \mapsto x + y$ on V and an operation of F on V:

$$F \times V \to V, \qquad (a, x) \mapsto ax.$$

These operations satisfy a whole bunch of axioms (VS 1–VS 8 in the book).

We refer to V with its two additional structures simply by the letter V.

Examples

An example that we all know: for each $n \geq 1$, the set $F^n = \{ (c_1, \ldots, c_n) \mid c_i \in F \text{ for } 1 \leq i \leq n \}$ becomes an F-vector space with the operations

$$(c_1, \ldots, c_n) + (d_1, \ldots, d_n) = (c_1 + d_1, \ldots, c_n + d_n)$$

and

$$a \cdot (c_1, \ldots, c_n) = (ac_1, \ldots, ac_n)$$

(componentwise addition and multiplication by elements of F).

The 0-vector space is the set $V = \{0\}$ with the obvious operations: $a \cdot 0 = 0$ for all $a \in F$, $0 + 0 = 0$. It is true somehow that $\{0\} = F^n$ when $n = 0$.

Warning: I sometimes write simply 0 for the 0-vector space. Our authors are careful to write $\{0\}$ instead. I like my shorthand, but don't want to force it on anyone.

Another example: the set of $m \times n$ matrices

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

again with component-wise addition and the obvious multplication by elements of F. This example is visibly a re-labeling of F^{mn}. This set is called $M_{m\times n}(F)$ in the book.

We can take V to be the space of all polynominals over F (of any degree) or the space of polynomials of degree $\leq n$ for a fixed non-negative integer n. These spaces are called $\wp(F)$ and $\wp_n(F)$, respectively.

Since a polynomial of degree $\leq n$,

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0,$$

is just a string of $n+1$ numbers, $\wp_n(F)$ is a suave way of writing F^{n+1}.

If S is a set, we can take V to be the set of functions $f : S \to F$, and define the operations in a pointwise manner. Thus $f+g$ is the function $(f+g)(s) = f(s)+g(s)$, and af is the function taking s to $af(s)$. If $S = \{1, 2, \ldots, n\}$, we get F^n.

An interesting variant is to take V instead to be the set of functions $S \to F$ that have only a finite number of non-zero values. This is a genuine variant if S is infinite.

For example, if S is the set of natural numbers (i.e., non-negative integers), the “variant” V that we have defined is the set of sequences $c_0, c_1, c_2, \ldots$ such that $c_m = 0$ for m sufficiently large. Such sequences are really the same thing as polynomials. Hence the V that you get in this case is the F-vector space of polynomials over F.

If you remove the requirement that c_m be zero for large m, you get “formal power series” (non-terminating polynomials) such as $1+x+x^2+x^3+\cdots$.

The axioms

First, V is an abelian group under addition:

- We have $x + y = y + x$ for $x, y \in V$;
- For $x, y, z \in V$, $x + (y + z) = (x + y) + z$;
- There is a (unique) $0 \in V$ such that $x + 0 = x$ for all $x \in V$;
- For each $x \in V$, there is a (unique) $-x \in V$ such that $x + (-x) = 0$.

We should all know the proof that 0 is unique and that additive inverses are unique. For the first, assertion, suppose that 0 and $0'$ both play the role of zero. Then $0 + 0'$ is both 0 and $0'$. Hence $0 = 0'$!

If y and z are additive inverses for x, then

$$y = y + 0 = y + (x + z) = (y + x) + z = 0 + z = z.$$

Next, two axioms about the action of F on V:

- For all $x \in V$, $1 \cdot x = x$;

- For $a, b \in F$ and $x \in V$, $a(bx) = (ab)x$.

Finally, there are two "distributive laws": $a(x+y) = ax + ay$ and $(a+b)x = ax + bx$, valid for $a, b \in F$, $x, y \in V$.

The deal about the axioms is that you want them to be easy to check but want them also to be powerful enough to imply familiar-looking statements that had better be true for us to preserve our sanity.

For example, we want $0 \cdot x = 0$ and $-(-x) = x$ for all $x \in V$, $(-a)x = a(-x) = -ax$ for $a \in F$, $x \in V$, etc., etc.

Let's prove something. Take a vector x in a vector space V. Consider $0 \cdot x$. We'd be amazed if $0x$ turned out to be anything other than 0 (the 0-vector).

How do we actually prove that $0x = 0$???

For $x \in V$, it's true that $0x = (0+0)x = 0x + 0x$ because $0+0 = 0$ in F and because of the distributive laws. Let $z = 0x$. Then $z = z+z$. We can add $-z$ to both sides and get $0 = z + (-z) = (z + z) + (-z) = z + (z + (-z)) = z + 0 = z$, so $z = 0$.

In Day One in math courses, we feel like idiots because we are proving assertions that we think should be obvious. After a couple of lectures, this is no longer true—we are proving hard things.

A psychologist named Nalini Ambady gave students three 10-second soundless videotapes of a teacher lecturing. Then she asked the students to rate the teacher. Their ratings matched the ratings from students who had taken the teacher's course for an entire semester. Then she cut the videotape back to two seconds and showed it to a new group. The ratings still matched those of the students who'd sat through the entire term.

—Review of Malcolm Gladwell's "Blink: Hunch Power"