Math 110

PROFESSOR KENNETH A. RIBET

Final Examination

May 10, 2010

11:30AM–2:30 PM, 10 Evans Hall

Please put away all books, calculators, and other portable electronic devices—anything with an ON/OFF switch. You may refer to a single 2-sided sheet of notes. For numerical questions, *show your work* but do not worry about simplifying answers. For proofs, write your arguments in complete sentences that explain what you are doing. Remember that your paper becomes your only representative after the exam is over. Please turn in your exam paper to your GSI when your work is complete.

The point values of the problems were 12, 7, 7, 8, 8, 8 for a total of 50 points.

1. Label each of the following statements as TRUE or FALSE. Along with your answer, provide a clear justification (e.g., a proof or counterexample).

a. Each system of n linear equations in n unknowns has at least one solution.

Obviously false: for example we could have the system $x + y = 1$, $x + y = 0$, which is clearly inconsistent (no solutions).

b. If A is an $n \times n$ complex matrix such that $A^* = -A$, every eigenvalue of A has real part 0.

Because $A^* = -A$, A commutes with its adjoint and thus is diagonalizable in an orthonormal basis. In this basis, we compute the adjoint by conjugating the elements on the diagonal. These elements must be the negatives of their conjugates, which implies that they are indeed purely imaginary. So the answer is “true”!

c. If W and W' are 5-dimensional subspaces of a 9-dimensional vector space V, there is at least one non-zero vector of V that lies in both W and W'.

Yes, this is true. A fancy way to see this is to consider the linear transformation $W \times W' \longrightarrow V$ taking a pair (w, w') to $w - w'$. Because $W \times W'$ has dimension $5+5 = 10$ and V has dimension $9 < 10$, there must be a non-zero element (w, w') in the null space of this map. We then have $w - w' = 0$, i.e., $w = w'$. Because w is in W and w' in W' and because these elements are equal, w lines in the intersection $W \cap W'$. So the correct answer is "true."

d. If T is a linear transformation on $V = \mathbf{C}^{25}$, there is a T-invariant subspace of V that has dimension 17.

Again, this is true: Schur's theorem tells you that there is an orthonormal basis of V in which T is upper-triangular. The span of the first 17 elements of this basis will be a T-invariant subspace of V of dimension 17.

2. Let T be a linear transformation on an inner-product space. Show that T^*T and T have the same null space.

This problem was done in class toward the end of the semester. If $Tv = 0$, then of course $T^*Tv = 0$ as well. The problem is to prove that if $T^*Tv = 0$, then already $Tv = 0$. However, if $T^*Tv = 0$, then $\langle T^*Tv, v\rangle = 0$. Using the definition of "adjoint," we convert the inner product to $\langle Tv, Tv\rangle$. Since this quantity is 0, the vector Tv must be zero (in view of the definition of an inner-product space).

3. Let $T : V \longrightarrow W$ be a linear transformation between finite-dimensional vector spaces over a field F. Show that there is a subspace X of V such that the restriction of T to X is 1-1 and has the same range as T.

If we admit the rank–nullity theorem, then we can do this problem in the following fairly brain-dead way. Choose a basis $v_1, \dots, v_m$ for the null space of T and extent this basis to a basis $v_1, \dots, v_n$ of V. Let X be the span of the last $n - m$ vectors in this basis. Clearly, the range of the restriction of T to X is the same as the range of T; indeed, the range of T is the set of all vectors $T(a_1v_1 + \cdots + a_nv_n)$, which is the set of all vectors $T(a_{m+1}v_{m+1} + \cdots + a_nv_n)$, i.e., the range of the restriction. Since the range has dimension $n - m$, which is the dimension of X, the restriction has to be 1-1.

A more enlightened way to do this is to redo the proof of the rank–nullity theorem.

4. Suppose that T is a self-adjoint operator on a finite-dimensional real vector space V and that $S : V \longrightarrow V$ is a linear transformation with the following property: every eigenvector of T is also an eigenvector of S. Show that there is a basis of V in which both T and S are diagonal. Conclude that S and T commute.

Choose an orthonormal basis of V in which T is diagonal. The elements of this basis must be eigenvectors of T and thus will be eigenvectors of S. Accordingly, S (as well as T) is diagonal in this basis. Since diagonal matrices commute with each other, S and T commute.

5. Use mathematical induction and the definition of the determinant to show for all $n \times n$ complex matrices A that the determinant of the complex conjugate of A is the complex conjugate of $\det A$. (The "complex conjugate" of a matrix A is the matrix whose entries are the complex conjugates of the entries of A.)

Let B be the complex conjugate of A. Work by induction as instructed. For the 1×1 case, A and B have one entry each, and these entries are conjugates of each other. The determinant of a 1×1 matrix is just the single element in the matrix, so we're good. For the induction step, we assume $n > 1$ and that the result is known for the $(n-1) \times (n-1)$ case. We have, by definition,

$$\det B = \sum_{j=1}^{n} (-1)^{1+j} b_{1j} \det \tilde{B}_{1j}.$$

In the sum, b_{1j} is the complex conjugate of a_{1j} by the definition of B. Also, $\det \tilde{B}_{1j}$ is the complex conjugate of $\det \tilde{A}_{1j}$ by the inductive assumption. (It's best to remark first that $\tilde{B}_{1j}$ is the complex conjugate of the matrix $\tilde{A}_{1j}$.) It follows that $\det B$ is the complex conjugate of $\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det \tilde{A}_{1j}$, as required.

6. Let W be a subspace of V, where V is a finite-dimensional vector space. Assume that W is a *proper* subspace of V (i.e., that it is not all of V). Show that there is a non-zero element of V^* that is 0 on each element of W. (A harder version of this problem was on the second midterm.)

Take a basis $v_1, \ldots, v_m$ of W and extend it to a basis $v_1, \ldots, v_n$ of V. Let $f_1, \ldots, f_n$ be the basis of V^* that's dual to $v_1, \ldots, v_n$. If $f = f_n$, f is non-zero but it's zero on $v_1, \ldots, v_m$ and therefore on W.