Math 110

Notes for the lecture on February 8, 2005

The lecture will be full of matrices and formulas. Here is a sketch of what I intend to talk about. Esepcially if you read this document before the lecture, you can take no notes or fewer notes than usual.

We work with vector spaces over a field F. Most are finite-dimensional. Take vector spaces V and W of finite dimension and suppose that we fix ordered bases β and γ of V and W, respectively. We have a beautiful association $\mathcal{L}(V, W) \to M_{m\times n}(F)$ given by $T \mapsto [T]_\beta^\gamma$. The source and target $\mathcal{L}(V, W)$ and $M_{m\times n}(F)$ are F-vector spaces in a natural way: we know how to add matrices and how to multiply matrices by scalars. Similarly, we know how to add linear transformations and multiply *them* by scalars—we discussed this on February 4 (Thursday). The first point is that $T \mapsto [T]_\beta^\gamma$ is a linear map (= linear tranformation) between vector spaces. This means that the sum of two Ts goes to the sum of their matrices and that $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for $a \in F$ and $T \in \mathcal{L}(V, W)$. These identities come from the definition of $[T]_\beta^\gamma$ that we gave on Thursday.

The claim is that $T \mapsto [T]_\beta^\gamma$ is an isomorphism of F-vector spaces. A linear map is said to be an isomorphism when it's invertible. This means that it's 1-1 and onto; we discussed invertible maps in class on Thursday. To see that the map is 1-1, we have to check that its null space is 0, i.e., that $[T]_\beta^\gamma = 0$ implies that $T = 0$. If $[T]_\beta^\gamma = 0$, then the construction of $[T]_\beta^\gamma$ shows that $T(v_j) = 0$ for all basis vectors $v_j \in \beta$. Since T is linear, $T = 0$. To see that the map is surjective (onto), we suppose that we are given a matrix $A = (a_{ij})$ in $M_{m\times n}(F)$. For each j, $1 \le j \le n$, let y_j be the element of W dictated to us by the jth column of A, namely $\sum_{i=1}^m a_{ij}w_i$, and then let T be the unique linear map $V \to W$ that sends v_j to y_j. (The existence of this map was discussed last week, especially on Thursday.) It is clear from the definition of $[T]_\beta^\gamma$ that this matrix coincides with the given matrix A.

One consequence is that the dimension of $\mathcal{L}(V, W)$ is mn, since mn is clearly the dimension of the space of $m \times n$ matrices over F. Remember that isomorphic vector spaces have equal dimensions.

By the way, here's a digression. If V has dimension n, then we get an isomorphism $F^n \xrightarrow{\sim} V$ by $(a_1, \dots, a_n) \mapsto \sum_{i=1}^n a_iv_i \in V$. If V and V' both have dimension n, then they are each isomorphic to F^n, so they are isomorphic to each other. Two finite-dimensional vector spaces are isomorphic if and only if they have equal dimensions. If V and W are vector spaces, we could write $V \sim W$ as an abbreviation for the statement "there exists some isomorphism from V to W." The statement means that V and W are isomorphic. The relation of being isomorphic is an equivalence relation. If $T : V \to W$ is a linear map between finite dimensional vector spaces of the same dimension and if $U : W \to V$ is a linear map such that $UT = 1_V$, then T and U are invertible and they are inverses of each other. End of digression.

The next theme is that of matrices of compositions. If we have $T : V \to W$ and $U : W \to Z$, we get UT (also written $U \circ T$), which means "T followed by U." It's a map $V \to Z$ and we know by now that it's a linear map. Formally, we could describe composition as a mapping

$$\mathcal{L}(W, Z) \times \mathcal{L}(V, W) \to \mathcal{L}(V, Z).$$

Assume that we have ordered bases

$$\alpha = \{v_1, \dots, v_n\}, \quad \beta = \{w_1, \dots w_m\}, \quad \gamma = \{z_1, \dots, z_d\}$$

of V, W and Z, respectively. The big claim is as follows:

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

On the right-hand side we have the product of a $d \times m$ matrix and an $m \times n$ matrix. On the left we have a $d \times n$ matrix. The dimensions are compatible with our identity being both meaningfull and true. It remains to compute things and check that everything works.

For the matrix T, we use familiar notation. For each $j = 1, \dots, n$, we write $Tv_j = \sum_{i=1}^m a_{ij} w_i$ and then say that $[T]_\alpha^\beta = A$ with $A = (a_{ij})$. Note that I write Tv_j instead of $T(v_j)$; omiting parentheses looks good here. Define $[U]_\beta^\gamma = B = (b_{ki})$ by the analogous formula; namely, write $Uw_i = \sum_{k=1}^d b_{ki} z_k$ for each i. Finally, introduce $[UT]_\alpha^\gamma = C = (c_{kj})$ by writing $UT(v_j) = \sum_k c_{kj} z_k$ for each j. Because of the linear independence of the z_k, the c_{kj} are the *unique* elements of F that satisfy these identities. What we need to check is that $C = BA$, which means that $c_{kj} = \sum_i b_{ki} a_{ij}$ for each k and j. It is enough to show that

$$UT(v_j) = \sum_k \sum_i b_{ki} a_{ij} \, z_k$$

for each j.

Now

$$UT(v_j) = U(\sum_i a_{ij} w_i) = \sum_i a_{ij} U(w_i) = \sum_i a_{ij} \sum_k b_{ki} z_k.$$

After rearranging the right-hand sum a tiny bit, we get the desired formula.

The formula that we have just proved actually specializes to a formula that we proved in the waning minutes of class on Thursday. This is interesting: Suppose that x is a vector in an n-dimensional vector space V. Consider the linear map $T : F \to V$ taking $a \in F$ to $ax \in V$. The vector space $F = F^1$ has the standard basis $\{1\}$. If we use this 1-element basis and an n-element basis $\beta = \{v_1, \dots, v_n\}$ of V, we get a matrix $[T]_{\{1\}}^\beta$, which is in fact an $n \times 1$ matrix—it's a column of length n. We see immediately from the definitions that $[T]_{\{1\}}^\beta = [x]_\beta$, where the right-hand side of the equation is the column that expresses x in terms of β. Now suppose that $U : V \to W$ is a linear transformation and that we have an ordered basis γ of W. Then we may write

$$[UT]_{\{1\}}^\gamma = [U]_\beta^\gamma [T]_{\{1\}}^\beta$$

by the general formula that we've proved. Now UT is the map $F \to W$ taking a to $U(x)$, so it's the analogue of T with $x \in V$ replaced by $Ux \in W$. Thus the displayed formula becomes $[Ux]_\gamma = [U]_\beta^\gamma [x]_\beta$, which is what we proved in class at the end of Thursday's lecture. (The map from V to W was called T instead of U in that lecture.)

We have seen that linear transformations give matrices and that every matrix of the right size comes from an element of $\mathcal{L}(V, W)$ when we have in the picture two spaces V and W with fixed bases β and γ. Another thing we can do is to take a matrix cold and not have any vector spaces around. Take $A \in M_{m\times n}(F)$ and notice that left-multiplication by A gives a map $F^n \to F^m$, $x \mapsto Ax$. Here, we think of F^n and F^m as spaces of column vectors. This map $F^n \to F^m$ is called L_A by our authors; the "L" could stand for "left" (as in multiplication on the left) or "linear." I'm pretty sure that they had "linear" in mind. It's easy to check that L_A is a linear map; this follows mainly from the distributive law for matrix multiplication, since we have to recognize that $A(x + x') = Ax + Ax'$. If β and γ are the standard bases on F^n and F^m, then $[L_A]_\beta^\gamma = A$. (You all probably remember that the standard basis vectors have one 1 and otherwise consist of 0s.)

Now we get to a super-important topic: change of basis. This really is just an application of compositions. Suppose that we have a map $T : V \to W$ between finite-dimensional F-vector spaces and that V and W have bases β and γ. The wrinkle now is that we assume that V has a second, alternative basis β'. We would like to compare $A := [T]_\beta^\gamma$ and $[T]_{\beta'}^\gamma$. Imagine that we understand how to write β' in terms of β. In other words, imagine that $\beta = \{v_1, \dots, v_n\}$ and $\beta' = \{v_1', \dots, v_n'\}$ and that we know how to express each v_j' as a linear combination of the v_i: $v_j' = \sum_i q_{ij} v_i$ for each j. The key insight is that the matrix $Q = (q_{ij})$ is nothing but $[1_V]_{\beta'}^\beta$; this really follows pretty directly from definitions in our setup. Using the formula for the matrix of a composite, we get

$$[T]_{\beta'}^\gamma = [T \circ 1_V]_{\beta'}^\gamma \; = [T]_\beta^\gamma [1_V]_{\beta'}^\beta = AQ.$$

When we change from β to β', we multiply $A = [T]_\beta^\gamma$ on the right by the change-of-basis matrix Q.

Assume finally that W has a second basis γ' and let $R = [1_W]_{\gamma'}^\gamma$ be the analogue of Q; it's the matrix that expresses the vectors in γ' in terms of those of γ. We see similarly that

$$[T]_{\beta'}^{\gamma'} = [1_W]_\gamma^{\gamma'} [T]_\beta^\gamma [1_V]_{\beta'}^\beta.$$

In the product on the right-side of the equation, we have already understood the second and third factors and need to identify the first factor $[1_W]_\gamma^{\gamma'}$. For various reasons that I'm not motivatived to type into this file, $[1_W]_\gamma^{\gamma'}$ is nothing but the inverse of the matrix $R = [1_W]_{\gamma'}^\gamma$. Thus the matrix of T with respect to the new bases β' and γ' is simply $R^{-1}AQ$ where R and Q are the transition matrices between the two βs and the two γs (as described above) and A is the matrix in the original bases.