Math 110

Notes for the lecture on February 15, 2005

We discussed duals of linear maps $T : V \to W$, where $V$ and $W$ are finite-dimensional. There's a natural map $W^* \to V^*$, which is composition with $T$: $g \in W^* \mapsto gT = g \circ T \in V^*$. This map is called $T^t$ by our authors; the "$t$" means "transpose." Suppose that $V$ is given with a basis $\beta$ and $W$ with a basis $\gamma$. Then we want to compare $[T]_\beta^\gamma$ and $[T^t]_{\gamma^*}^{\beta^*}$, where $\beta^*$ and $\gamma^*$ are the bases of $V^*$ and $W^*$ that are dual to $\beta$ and $\gamma$. The first matrix is an $m \times n$ matrix, while the second is an $n \times m$ matrix. The fundamental result is that the two matrices are *transposes* of each other. I reproduce here the computation that I made on a sheet of paper last week. See, alternatively, page 121 of the book.

Let the matrix of $T$ be $A$. In other words, suppose as usual that $Tv_j = \sum_{i=1}^m a_{ij} w_i$. Let the matrix of $T^t$ be $C$. We write $\beta^* = \{f_1, \ldots, f_n\}$ and $\gamma^* = \{g_1, \ldots, g_m\}$. Then the entries of $C$ are defined as follows: $T^t(g_j) = \sum_{k=1}^n c_{kj} f_k$. In this equation, each side is a linear functional $V \to F$. Evaluate the two sides on a basis vector $v_i$. The right-hand side yields $\sum_{k=1}^n c_{kj} f_k(v_i) = c_{ij}$ because of the usual Kronecker delta collapsing. The left-hand side gives $T^t(g_j)(v_i) = g_j(T(v_i)) = g_j(\sum_{k=1}^m a_{ki} w_k) = a_{ji}$, with the last equality following once again from Kronecker delta collapsing. We thus get $c_{ij} = a_{ji}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. This is the desired equality.

The next and final thing that we will discuss about duals right now is the fundamental isomorphism $\psi : V \xrightarrow{\sim} V^{**}$ when $V$ is a finite-dimensional vector space. This is mind-bending, but not hard in essence. What happens algebraically is that there is a natural injection $V \hookrightarrow V^{**}$ whenever $V$ is an $F$-vector space. If $V$ has finite dimension, then the source and target both have the same (finite) dimension, so the injection is an isomorphism. Here we use crucially that a 1-1 map between vector spaces of the same finite dimension is necessarily an isomorphism.

We move on to Chapter 3. This is an awkward part of the course because the material in Chapter 3 is important to further study but is hard to discuss in class for two reasons. The first reason is that most of you know in your fingers (from Math 54) how to perform the manipulations that are discussed by the authors. The second is that the proof that these manipulations work in general is tedious and unilluminating. It is remarkable how much one can prove by thinking about elementary row and column operations. Everyone should learn what the elementary matrices look like.