Ribet's Math 110 Second Midterm, problems and abbreviated solutions

Please put away all books, calculators, and other portable electronic devices—anything with an ON/OFF switch. You may refer to a single 2-sided sheet of notes. When you answer questions, write your arguments in complete sentences that explain what you are doing: your paper becomes your only representative after the exam is over.

All vector spaces are finite-dimensional over the field of real numbers or the field of complex numbers.

1. Suppose that T is an invertible linear operator on V and that U is a subspace of V that is invariant under T. If v is a vector in V such that $Tv \in U$, show that v is an element of U.

Quick solution: Let $u = Tv$. Because the restriction of T to U is invertible, there is a unique $v' \in U$ such that $Tv' = u$. Since $Tv' = Tv$ and T is invertible, we have $v' = v$. Hence we have $v \in U$.

2. Suppose that T is a linear operator on V and that V is an inner-product space. Let T^* be the adjoint of T. Show that 0 is an eigenvalue of T if and only if 0 is an eigenvalue of T^*.

Quick solution: This problem is the special case of problem 28 where we take $\lambda = 0$. In fact, if we can do this special case, then we get the full statement of problem 28 by replacing T by $T - \lambda I$. To say that 0 is an eigenvalue of an operator is to say that the operator is not invertible. Equivalently, this means that its range is smaller than V and also that its null space is non-zero. Thus T^* has 0 as an eigenvalue if and only if its null space is non-zero, and T has 0 as an eigenvalue if and only if its range is smaller than V. To see that these statements are equivalent, we can invoke part (a) of Proposition 6.46 on page 120. Specifically, let $U = T^*$. Then U is $\{0\}$ is and only if $U^\perp = V$. These statements follows from the equations $\{0\}^\perp = V$ (everything is perpendicular to 0), $V^\perp = \{0\}$ (only 0 is perpendicular to everything) and $(U^\perp)^\perp = U$ (6.33 on page 112).

3. Let T be a linear operator on V. Suppose that there is a non-zero vector $v \in V$ such that $T^3 v = Tv$. Show that at least one of the numbers 0, 1, -1 is an eigenvalue of T.

Quick solution: Because v is in the null space of $T^3 - T$, this operator is not invertible. However, it is the product $T(T-I)(T+I)$; note that the polynomial $x^3 - x$ factors as $x(x-1)(x+1)$! Because the product is non-invertible, at least one factor is non-invertible. To say that T is non-invertible is to say that 0 is an eigenvalue of T. To say that $T - I$ is non-invertible is to say that 1 is an eigenvalue of T. To say that $T + I$ is non-invertible is to say that -1 is an eigenvalue of T. 'null said.

4. Let U be a subspace of the inner-product space V, and let $P = P_U$ be the orthogonal projection of V onto U. [For $v \in V$, write $v = u + y$ with $u \in U$ and $y \in U^\perp$. Then $Pv = u$.] Show that $P = P^*$.

Quick solution: We need to establish the equality $\langle Pv, w\rangle = \langle v, Pw\rangle$ for $v, w \in V$. Let v and w be in V, and write $v = u + x$ as in the statement of the problem. Similarly, write $w = u' + x'$. Then we need to prove $\langle u, u' + x'\rangle = \langle u + x, x'\rangle$. However, the term on the left is $\langle u, u'\rangle + \langle u, x'\rangle = \langle u, u\rangle$ because u and x' are perpendicular. Similarly, the term on the right simplifies to $\langle u, u'\rangle$.