Math 110

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Final Examination

December 20, 2008

12:40–3:30 PM, 101 Barker Hall

Please put away all books, calculators, and other portable electronic devices—anything with an ON/OFF switch. You may refer to a single 2-sided sheet of notes. When you answer questions, write your arguments in complete sentences that explain what you are doing: your paper becomes your only representative after the exam is over. All vector spaces are finite-dimensional over the field of real numbers or the field of complex numbers (except for the space $\mathcal{P}(\mathbf{R})$ of all real polynomials, which occurs in the first problem).

Problem	Possible points
1	9 points
2	6 points
3	7 points
4	7 points
5	7 points
6	7 points
7	7 points
Total:	50 points

1. Exhibit examples of:

(**a.**) A linear operator $D : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ and a linear operator $\mathcal{I} : \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ such that $D\mathcal{I}$ is the identity but $\mathcal{I}D$ is not the identity.

(**b.**) A (non-zero) generalized eigenvector that is not an eigenvector.

(**c.**) A normal operator on a positive-dimensional real vector space whose characteristic polynomial has no real roots.

In the first part, "D" was intended to evoke differentiation and "$\mathcal{I}$" was intended to suggest integration (definition integration with constant term 0, for example). It's hard to give "answers" to these questions because different people will have different examples.

2. On the vector space $\mathcal{P}_2(\mathbf{R})$ of real polynomials of degree ≤ 2 consider the inner product given by

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

Apply the Gram–Schmidt procedure to the basis $(1, x, x^2)$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

When you apply the Gram–Schmidt progress to the sequence of polynomials $1, x, x^2, x^3, \ldots$, the resulting sequence of orthonormal polynomials is the sequence of Legendre polynomials. According to the Wikipedia article on Orthogonal Polynomials, the first three of them are 1, x and $(3x^2 - 1)/2$. I haven't done this calculation lately, but I'll be wading through 38 such calculations in the very near future.

3. Suppose that P is a linear operator on V satisfying $P^2 = P$ and let v be an element of V. Show that there are unique $x \in \operatorname{null} P$ and $y \in \operatorname{range} P$ such that $v = x + y$.

First, let $x = v - Pv$ and $y = Pv$. Then clearly $v = x + y$, and $y = Pv$ is in the range of P. Since $Px = Pv - P^2v = Pv - Pv = 0$, x is in the null space of P. Secondly, suppose that $v = x + y$ with $x \in \operatorname{null} P$ and $y \in \operatorname{range} P$. Then $Pv = Px + Py = 0 + Py = 0$. Further, $Py = y$ because y is in the range of P. Indeed, if $y = Pw$, then $Py = P(Pw) = P^2w = Pw = y$. Hence $Pv = y$, which implies that $x = v - y = v - Pv$. In other words, the x and y that we are dealing with are the ones that we knew about already. Conclusion: x and y are unique. Note: this was Exercise 21 of Chapter 5.

4. Let T be a linear operator on an inner product space for which $\operatorname{trace}(T^*T) = 0$. Prove that $T = 0$.

Choose an orthonormal basis for the space, and let $A = [a_{ij}]$ be the matrix of T in this basis. The matrix of T^* is the conjugate-transpose A^* of A. The matrix of T^*T is then A^*A; for each i, $i = 1, \ldots, n$, the (i,i)th entry of this matrix is $\displaystyle\sum_j \overline{a}_{ji} aji = \sum_j |a_{ji}|^2$. The trace of A^*A is thus $\displaystyle\sum_{i,j} |a_{ji}|^2$. Each term $|a_{ji}|^2$ is a non-negative real number. If the sum is 0, then each term is 0. This means that all a_{ij} are 0, i.e., that $A = 0$. If $A = 0$, then of course $T = 0$. Note: See problem 18 of Chapter 10, where a somewhat more sophisticated proof of the indicated assertion was contemplated by the author of our book.

5. If X and Y are subspaces of V with $\dim X \geq \dim Y$, show that there is a linear operator $T : V \to V$ such that $T(X) = Y$.

Let $d = \dim Y$ and let $n = \dim V$. Choose a basis $(v_1, \ldots, v_e)$ of X; note that $e \geq d$ by hypothesis. Complete this basis to a basis $(v_1, \ldots, v_n)$ of V. Choose a basis $(y_1, \ldots, y_d)$ of Y. We define $T : V \to V$ so that $T(v_i) = y_i$ for $i = 1, \ldots, d$ and $T(v_i) = 0$ for $i > d$. Namely, if $v = \displaystyle\sum_{i=1}^{n} a_i v_i$, we set $Tv = \displaystyle\sum_{i=1}^{d} a_i y_i$. The range of T is then the span of the y_i, which is Y. Already, however, the span of $(v_1, \ldots, v_d)$ is mapped onto Y by T. Thus X is mapped onto Y by T.

6. Let N be a linear operator on the inner product space V. Suppose that N is both normal and nilpotent. Prove that $N = 0$. [The case $F = \mathbf{C}$ will probably be easier for you. Do it first do ensure partial credit.]

In the complex case, we can invoke the spectral theorem and find an orthonormal basis in which N has a diagonal matrix representation. Since some power of N is 0, the diagonal entries are all 0. Hence $N = 0$, as required. In the real case, the required assertion follows from the statement of Exercise 24 of Chapter 8, or—even better—from Exercise 7 of Chapter 7. (You have Axler's solution to that exercise.) Alternatively, we can argue (cheat) as follows: choose an orthonormal basis for the space, so that N becomes a nilpotent matrix of real numbers that commuters with its transpose. We can think of this as a nilpotent matrix of complex numbers that commutes with its adjoint. Such a matrix corresponds to a complex normal nilpotent operator, which we already know to be 0. Hence it's 0.

7. Suppose n is a positive integer and $T : \mathbf{C}^n \to \mathbf{C}^n$ is defined by

$$T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n);$$

in other words, T is the linear operator whose matrix (with respect to the standard basis) consists of all 1's. Find all eigenvalues and eigenvectors of T.

This was Exercise 7 of Chapter 5. You should have a solution available to you.