Math 110 Professor K. A. Ribet

Midterm Exam September 29, 2003

Please put away all books, calculators, electronic games, cell phones, pagers, .mp3 players, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Please write your name on each sheet of paper that you turn in; don't trust staples to keep your papers together. Explain your answers in full English sentences as is customary and appropriate. Your paper is your ambassador when it is graded.

All problems were worth 10 points.

1. Suppose that F is the field of rational numbers. Let $V = \mathbf{P}_{100}(F)$ be the F-vector space consisting of polynomials over F of degree ≤ 100. Let $T = \frac{d}{dx}\colon V \to V$ be the differentiation operator $\sum_{i=0}^{n} a_i x^i \mapsto \sum_{i=1}^{n} i a_i x^{i-1}$. Find the nullity and the rank of T.

The kernel (= null space) of T is the 1-dimensional vector space consisting of constant polynomials. Hence $n(T) = 1$ and $r(T) = \dim(V) - n(T) = 101 - 1 = 100$.

Suppose now instead that F is the field Z_5 consisting of the integers 0, 1, 2, 3 and 4 mod 5. What are the nullity and the rank in this case?

The null space includes some non-constant polynomials; x^5 is a good example. (Note that $5x^4 = 0x^4 = 0$.) On reflection, you see that $f(x)$ has derivative equal to 0 if and only if f contains only monomials x^i where i is divisible by 5. Hence $N(T)$ is generated by 1, x^5, x^{10}, $\ldots x^{100}$; it has dimension 21. Thus $r(T) = 101 - 21 = 80$ in this situation.

2. Let V and W be vector spaces over F, with V finite-dimensional. Let X be a subspace of V. Establish the surjectivity ("onto-ness") of the natural map $\mathcal{L}(V, W) \to \mathcal{L}(X, W)$ that takes a linear transformation $T\colon V \to W$ to its restriction to X.

One has to show that every linear map $X \to W$ can be extended to V. For this, it is convenient to choose a basis of V by beginning with a basis $x_1, \ldots, x_d$ of V

and then completing it to a basis $x_1, \dots, x_d; v_{d+1}, \dots, v_n$ of V. If $T\colon X \to W$ is a linear transformation, one can define $U\colon V \to W$ by the formula

$$U(\sum a_i x_i + \sum b_j v_j) := T(\sum a_i x_i).$$

This makes sense because each vector of V may be written uniquely in the form $\sum a_i x_i + \sum b_j v_j$. It is routine to check that U is a linear map and that U coincides with T on X.

3. Let V be a finite-dimensional vector space over F. Let V^* be the vector space dual to V. Let $T\colon V^* \to F$ be a linear map. Show that there is a vector $x \in V$ such that $T(f) = f(x)$ for all $f \in V^*$

This problem amounts to the surjectivity of the natural map $\psi\colon V \to V^{**}$ that is discussed in §2.6 of the book. In the notation of that section, we are trying to show that there is an $x \in V$ so that $T = \hat{x}$. I am crossing my fingers and hoping that people will explain something of what is going on and not say simply that the wanted result was proven in class or say that it follows from Theorem 2.26 of the book. Take a basis $v_1, \dots, v_n$ for V and let f_i be the vectors in the basis dual to $\{ v_1, \dots, v_n \}$. Let $a_i = t(f_i)$ for $i = 1, \dots, n$ and set $x = a_1 v_1 + \cdots + a_n v_n$. A quick computation should show that this x has the desired propoerty.