Math H110 Professor K. Ribet

Homework assignment #12, due November 14

• I'd like to ask you to do the following problem, which George Bergman is assigning to his Math 110 students. It amounts to establishing the diagonalizability of real self-adjoint operators without moving up first to the field of complex numbers. To set the stage, we consider a non-zero inner-product space $V$ of finite dimension $n$ over the field $\mathbf{R}$ of real numbers. We know very well what $V$ looks like; indeed, if we choose an orthonormal basis of $V$, then $V$ becomes isomorphic to $\mathbf{R}^n$ in such a way that the inner product on $V$ becomes the standard inner product. Without much loss of face, we can suppose if we want that $V = \mathbf{R}^n$ and that the inner product is the standard one. In particular, the set $S := \{\, x \in V \mid \|x\| = 1 \,\}$ is a non-empty closed and bounded set—it's nothing other than the unit $(n-1)$-sphere when $V = \mathbf{R}^n$. Notice that every non-zero vector in $V$ may be written uniquely as the product of a non-zero real number and an element of $S$.

Calculus tells us that every continuous real-valued function on $S$ achieves a maximim value on $S$. In particular, if $T\colon V \to V$ is a linear map, the function $s \mapsto \langle s, T(s)\rangle$ on $s$ achieves a maximum at some point $p$ of $S$.

**(a)** Supppose that $x$ and $y$ are points on $S$ such that $\langle x, y\rangle = 0$. Show that $\cos(t)\cdot x + \sin(t)\cdot y$ lies on $S$ for all $t \in \mathbf{R}$.

**(b)** Choose $p$ in $S$ as above, i.e., so that $\langle p, T(p)\rangle \geq \langle s, T(s)\rangle$ for $s \in S$. Take $y \in S$ with $\langle p, y\rangle = 0$ and consider $f(t) := \langle\, \cos(t)\cdot p + \sin(t)\cdot y\,,\, T\,(\cos(t)\cdot p + \sin(t)\cdot y)\,\rangle$. This function has a maximum at $t = 0$, and therefore $f'(0) = 0$. Re-write this equation as a relation involving inner products.

**(c)** Now suppose that $T$ is self-adjoint. Show that we have $\langle p, T(y)\rangle = 0$ in the situation of (b). Let $W$ be the 1-dimensional subspace of $V$ that is spanned by $p$ and let $W'$ be the space of vectors that are orthogonal to $p$. Show that $W'$ is $T$-invariant (i.e., that $T(x) \in W'$ for all $x \in W'$) and then conclude that $W$ is $T$-invariant. Notice that this means that $p$ is an eigenvector!

**(d)** Parlay the argument of (c) into an inductive proof that $V$ has an orthonormal basis of vectors that are eigenvectors for $T$.

• §6.5: 4, 15, 16, 17, 31, 32

Here's a discussion of Problem 16 on page 394, which some students asked me about last week. When they came into my office, I had no clue how to find an example. Let $F$ be $\mathbf{R}$, the field of real numbers. Let $V$ be the space of polynomials over $\mathbf{R}$ in $t$ and $t^{-1}$. Such polynomials are finite sums $\sum a_i t^i$, where the $a_i$ are real numbers and the indices $i$ are integers. We think of them as "infinite sums" taken over the set of all integers $i$ in which only finitely many of the terms are actually non-zero. Notice that some of the $i$ for which $a_i$ is non-zero may be negative integers. A basis of $V$ is the set of monomials $1, t, t^{-1}, t^2, t^{-2}, \ldots$. We define an inner product on $V$ by $\langle \sum a_i t^i, \sum b_i t^i\rangle := \sum a_i b_i$. This is just a harmless-looking variant of the standard inner product on Euclidean space $\mathbf{R}^n$. Note again that the sum defining the inner product is formally an infinite sum but that there are only finitely many non-zero terms in in! Let $U\colon V \to V$ be the linear map "multiplication by $t$"; this map has an evident inverse, namely multiplication by $t^{-1}$. Also, it's clearly unitary as we see from the definition of the inner product on $V$. Now let $W \subseteq V$ be the subspace consisting of the usual polynomials in $t$ (ones with no negative powers of $t$). This space is clearly invariant under $U$ since multiplication of a polynomial by $t$ yields another polynomial. The subspace $W^{\perp}$ is the space generated by $t^{-1}, t^{-2}, t^{-3}, \ldots$, i.e., the space of polynomials with only negative powers of $t$. This space is not stable under $U$; for example, $U(t^{-1}) = 1$ is not in $W^{\perp}$.