Math 110 Professor K. A. Ribet

Midterm Exam September 26, 2002

Please put away all books, calculators, digital toys, cell phones, pagers, PDAs, and other electronic devices. You may refer to a single 2-sided sheet of notes. Please write your name on each sheet of paper that you turn in. Don't trust staples to keep your papers together.

The symbol "$\mathbf{R}$" denotes the field of real numbers. In this exam, "0" was used to denote the vector space $\{0\}$ consisting of the single element 0.

These solutions were written by Ken Ribet. Sorry if they're a little terse. If you have a question about the grading of your problem, see Ken Ribet for problems 1–2 and Tom Coates for 3–4.

1. (9 points) Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear transformation whose matrix with respect to the standard bases is $A = \begin{pmatrix} -1 & -1 & 7 & 5 \\ 1 & 0 & -5 & -3 \\ 0 & -1 & 2 & 2 \end{pmatrix}$. (In the book's notation, $T = L_A$.)

Find bases for (i) the null space and (ii) the range of T.

The null space consists of quadruples (x, y, z, w) satisfying three equations of which the first two are $-x-y-7z+5w = 0$ and $x-5z-3w = 0$. It turns out that the third equation is the sum of the first two, so we can forget it. (If you don't notice this circumstance, you'll still get the right answer.) Replace the first equation by the sum of the first two, leaving the second alone. We get the two equations

$$\begin{aligned} -y + 2z + 2w &= 0 \\ x \qquad - 5z - 3w &= 0. \end{aligned}$$

The interpretation is that z and w can be chosen freely, and then x and y are determined by z and w. If we take $z = 1$, $w = 0$, we get the solution $(5, 2, 1, 0)$. With the reverse choice, we get the solution $(3, 2, 0, 1)$. These form a basis for the null space.

Once we know that the null space has dimension 2, we deduce that the range has dimension 2 as well. In fact, it consists of the space of triples (a, b, c) with $c = a + b$. A basis would be the set containing $(1, 0, 1)$ and $(0, 1, 1)$. Of course, there are other correct answers: bases aren't unique! Let me stress that $R(T)$ lies in 3-space. If your answer consists of vectors in $\mathbf{R}^4$, you've messed up.

Note from Ribet: An answer that just has a bunch of numbers with no explanation as to what is going on is very unlikely to receive full credit. You need to tell the reader (me, in this case) what you are doing.

2. (9 points) Let V be a vector space over a field F. Suppose that $v_1, \ldots, v_n$ are elements of V and that $w_1, \ldots, w_n, w_{n+1}$ lie in the span of $\{ v_1, \ldots, v_n \}$. Show that the set $\{ w_1, \cdots, w_{n+1} \}$ is linearly dependent.

Let W be the span of $\{ v_1, \ldots, v_n \}$. Then W is generated by n elements, so its dimension d is at most n (for example, by Theorem 1.9 on page 42). If the vectors w_i were linearly independent, the set $\{ w_1, \cdots, w_{n+1} \}$ could be extended to a basis of W. This is impossible because all bases of W have d elements.

3. (10 points) Let W_1 and W_2 be subspaces of a finite-dimensional F-vector space V. Recall that $W_1 \times W_2$ denotes the set of pairs (w_1, w_2) with $w_1 \in W_1$, $w_2 \in W_2$. This product comes equipped with a natural addition and scalar multiplication:

$$(w_1, w_2) + (w_1', w_2') := (w_1 + w_1', w_2 + w_2'), \qquad a(w_1, w_2) := (aw_1, aw_2).$$

This addition and scalar multiplication make $W_1 \times W_2$ into an F-vector space. *(There was no requirement or expectation that students verify this point.)*

(1) Check that the map

$$T : W_1 \times W_2 \to V, \quad (w_1, w_2) \mapsto w_1 + w_2$$

is a linear transformation.

This is fairly routine. For example, $T(a(w_1, w_2)) = T(aw_1, aw_2) = aw_1 + aw_2 = a(w_1 + w_2) = aT(w_1, w_2)$. A similar computation shows that T of a sum is the sum of the T's.

(2) Prove that $N(T) = 0$ if and only if $W_1 \cap W_2 = 0$.

The space $N(T)$ consists of pairs (w_1, w_2) with $w_1 + w_2 = 0$. Hence w_2 is completely determined by w_1 as its negative. The wrinkle is that w_1 has to be in W_1 while $w_2 = -w_1$ has to be in W_2. Thus w_1 has to be in both W_1 and W_2, i.e., in the intersection of the two spaces. The null space $N(T)$ is in 1-1 correspondence with $W_1 \cap W_2$, with $(w_1, w_2) \in N(T)$ corresponding to $w_1 \in W_1 \cap W_2$ and $w \in W_1 \cap W_2$ corresponding to $(w, -w) \in N(T)$. In particular, $N(T) = 0$ if and only if $W_1 \cap W_2 = 0$.

(3) Show that $\dim(W_1 \cap W_2) \geq \dim W_1 + \dim W_2 - \dim V$.

Consider $T : W_1 \times W_2 \to V$. We have

$$\dim(W_1 \times W_2) = \dim N(T) + \dim R(T) = \dim(W_1 \cap W_2) + \dim R(T).$$

Here, I have used the fact that the identification between $N(T)$ and $W_1 \cap W_2$ that we discussed above is a linear identification—one that respects addition and scalar multiplication. Hence it preserves dimension. For this problem, we have to use another fact, namely that $\dim(W_1 \times W_2))$ is the sum of the dimensions of W_1 and W_2. This follows from the *fact that we get a basis for the product by taking the union of* $\beta_1 \times \{0\}$ *and* $\{0\} \times \beta_2$, *where the* β_i *are bases of the* W_i $(i = 1, 2)$. *This fact gives*

$$\dim(W_1 \times W_2) = \dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim R(T) \leq \dim(W_1 \cap W_2) + \dim V.$$

The desired inequality follows.

4. (12 points) Label the following statements as being *true* or *false*. For each statement, explain your answer

(1) The span of the empty set is the empty vector space.

There is no empty vector space! The span of the empty set is $\{0\}$.

(2) If v is a vector in a vector space V that has more than two elements, then V is spanned by the set $S = \{ w \in V \mid w \neq v \}$.

The span of S *contains* S, *so it contains all* w *different from* v. *Does it contain* v *as well? Yes, indeed: choose* $w \in V$ *different from 0 and* v; *this choice is possible because* V *has more than two elements. Write* $v = (v - w) + w$. *The vectors* $v - w$ *and* w *are both in* S: *neither is* v. *Hence* v *is in the span of* S. *Our conclusion is that the span of* S *contains all of* V, *so the assertion is true.*

(3) Suppose that $T : V \to W$ is a linear transformation between finite-dimensional $\mathbf{R}$-vector spaces. If $\dim V > \dim W$ and w lies in the range of T, then there are infinitely many $v \in V$ such that $T(v) = w$.

If w *lies in the range of* T, *then there is some* $v \in V$ *such that* $T(v) = w$. *Fix this* v, *and notice that* $T(v') = w$ *if and only if* $T(v' - v) = 0$. *Thus the set of* v' *mapping to* w *is the set of* $v + x$ *where* x *runs over* $N(T)$. *Thus there are infinitely many elements of* V *that map to* w *if and only if* $N(T)$ *is infinite. Since we are working over the field of real numbers, which is an infinite field,* $N(T)$ *is infinite if and only if it is non-zero. Were* $N(T)$ *zero, we would have* $\dim R(T) = \dim V$. *However,* $\dim R(T)$ *is at most* $\dim W$, *which is less than* $\dim V$. *Accordingly,* $N(T)$ *must be non-zero. The assertion is true.*

(4) Suppose that $T : V \to W$ is a linear transformation between finite-dimensional $\mathbf{R}$-vector spaces. If $\dim V < \dim W$ and w lies in the range of T, then there is exactly one $v \in V$ such that $T(v) = w$.

The assertion being made is not necessarily true; thus the best answer to the question is "false." For example, V *could be 1-dimensional,* W *could be 10-dimensional,* T *could be identically 0, and* w *could be 0. The set of* v *mapping to* w *would then be the entire vector space* V, *which is infinite in the situation that we're contemplating.*