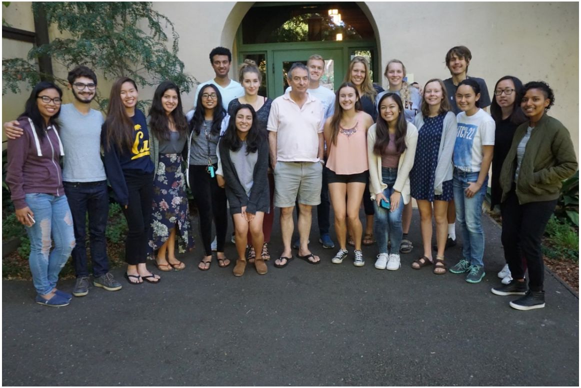
More on infinite series
Antiderivatives and area

Math 10A

September 28, 2017

The eighth Math 10A breakfast was on Monday:



There are still slots available for the October 4 breakfast (Wednesday, 8AM), and there's a pop-in lunch on Friday at noon.

Today we'll discuss a few infinite series examples and then move on to areas and the antiderivative.

Review of the ratio test

Take an infinite series $\sum_{n=1}^{\infty} a_n$ with the a_n non-zero (except perhaps for the first few). The a_n are allowed to be negative. Suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists. Then

- If the limit is less than 1, the series converges.
- If the limit is greater than 1, the series diverges (and in fact its nth term doesn't approach 0).

If the limit is 1, the test says nothing about convergence.

Use the ratio test to decide whether or not $\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{2^n}$ is convergent.

The ratios $\left|\frac{a_{n+1}}{a_n}\right|$ converge to $\frac{1}{2} < 1$, so the series converges.

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Use the ratio test to decide whether or not $\sum_{n=1}^{\infty} \frac{10^n}{n!}$ is convergent.

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Power series

For which numbers x is the series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = 1 + x + x^2/2 + x^3/3 + x^4/4 + \cdots$$

convergent?

Set $a_n = \frac{x^n}{n}$; then $\left|\frac{a_{n+1}}{a_n}\right| = |x|\frac{n}{n+1} \to |x|$. The series diverges for $|x| > 1$ (nth term doesn't approach 0) and converges (absolutely) for x in $(-1, 1)$.

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When $x = 1$, the series is the harmonic series

$$1 + 1/2 + 1/3 + \cdots .$$

It's divergent.

When $x = -1$, the series is the negative of the "alternating harmonic series"

$$1 - 1/2 + 1/3 - 1/4 + 1/5 - \cdots .$$

This latter series (and thus the original one, which is the negative of the series that was just written down) is convergent by a criterion known as the...

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Alternating series test

Suppose that a_1, a_2, a_3,... are positive numbers that

- decrease ($a_1 \geq a_2 \geq a_3 \geq a_4 \cdots$) and
- approach 0 ($a_n \downarrow 0$).

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots$$

converges.

The proof is by a diagram—let's move to the document camera for this.

Miscellaneous infinite series example

Decide whether the infinite series $\sum_{n=1}^{\infty} \frac{n}{n+2}$ is convergent? If it is convergent, can you find the sum?

The nth term approaches 1, which isn't 0, so the series diverges.

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It's just the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ that has been deprived of its first term. That change doesn't affect divergence or convergence (to be discussed).

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Antiderivatives

An antiderivative of a function f is a function F whose derivative is f:

$$F'(x) = f(x).$$

For example, x^3 is an antiderivative of $3x^2$.

And $-\cos(x)$ is an antiderivative of $\sin(x)$.

An antiderivative of x^4 is $\frac{x^5}{5}$.

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Finding antiderivatives in these examples was done by guesswork. We know lots of derivatives and can often figure out a function F such that $F' = f$.

Sometimes we have to multiply or divide by a constant, as in the last example.

An important principle is that two functions with the same derivative differ by a constant:

$$F' = f \text{ and } G' = f \Longrightarrow G = F + C.$$

The reason is that $F - G$ has derivative 0 and thus can't rise or fall. It's just a constant. Sad!!

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People say:

The antiderivative of x^{-6} is $\frac{-1}{5}x^{-5} + C$.

They write:

$$\int x^{-6}\, dx = \frac{-1}{5}x^{-5} + C.$$

Pinning down the constant—initial values

Suppose that F is defined for positive numbers and satisfies:

- $F'(x) = \frac{1}{x}$ for $x > 0$,
- $F(1) = 0$.

Do we then know F exactly?

Because $\frac{d}{dx} \ln x = \frac{1}{x}$, $F(x) = \ln x + C$ for some constant C.

Set $x = 1$. Then

$$0 = F(1) = \ln(1) + C = 0 + C,$$

so $C = 0$. Therefore, F is the natural log function, period. The word "period" means that no constant needs to be added or discussed. Case closed. (This is as far as we got in class on Thursday.)

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Connection with area

[We'll start here on Tuesday, October 3.]

To start, imagine that f is a positive (continuous) function defined on $[a, b]$ (the closed interval of numbers from a to b, with $a \leq b$). We can make sense out of the

area under the curve $y = f(x)$ between a and b.

Area is usually defined by approximation: stuff lots of rectangles into the region whose areas you want to calculate, sum up the areas of the rectangles and take a limit as the rectangles become thiner and thiner (and more and more numerous).

Allow me to make a crude sketch on the document camera.

Area as a new, mysterious function

If you think of a as fixed and b as varying, the area under $y = f(x)$ from a to b becomes a function of b:

$A(b)$ = the area under the graph of f between a and b,

$A(x)$ = the area under the graph of f between a and x.

One thing that you can say about this function is that $A(a) = 0$: there's no positive area between a and a.

The Fundamental Theorem of Calculus

The fundamental theorem states:

$$A' = f.$$

In words: the derivative of the area function is the function under which you compute the area.

Area is an antiderivative of f.

The area function satisfies $A' = f$ and $A(a) = 0$. These two conditions tell you exactly what function area is.

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The log example

Take $f(x) = \dfrac{1}{x}$ and let $a = 1$. Then the area function $A(b)$ represents the area under $y = 1/x$ from 1 to b.

This function satisfies $A'(x) = 1/x$ and $A(1) = 0$.

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Why is the fundamental theorem true?

The derivative $A'(b)$ is the limit as $h \to 0$ of the fraction

$$\frac{A(b+h) - A(b)}{h}.$$

The numerator represents the area under $y = f(x)$ between $x = a$ and $x = a + h$. If h is small, this sliver of area is roughly a rectangle with base h and height $f(b)$. Hence

$$\frac{A(b+h) - A(b)}{h} \approx \frac{f(b) \cdot h}{h} = f(b).$$

As $h \to 0$, the approximation becomes better and better. As you take the limit it yields an equality:

$$A'(b) = \lim_{h \to 0} \frac{A(b+h) - A(b)}{h} = f(b).$$

I will illustrate what's going on with a crude hand-drawn diagram on the document camera. That's if we get as far as these last slides during the class meeting on Friday.

If we don't—which is what I'm guessing—we'll discuss this topic on Tuesday, October 3.

That's the day before the October 4 breakfast for which there are still several slots available.

See you next week!

Or at tomorrow's pop-in lunch!!