Infinite series

Math 10A

September 21, 2017

Schedule

- The first midterm is next Tuesday, here in class.
- Midterm rules: no devices of any kind, but you can bring a two-sided single sheet of $8\frac{1}{2} \times 11$ notes.
- Discussion sections will meet next Tuesday.
- HW #4 is due today.
- HW #5 will be due on Tuesday, October 3.
- The next quiz will be on Thursday, September 28.

The seventh Math 10A breakfast was yesterday:

Note that we had a distinguished photobomber in the photo.

A breakfast has been created for Wednesday, October 4. Sent email to sign up for that date!

Pop-in lunch on Friday at noon.

Today's class meeting

We'll discuss infinite series, mainly through examples.

The slides that follow—those about the repeating decimal—will be skipped in class. Please go through them on your own (informal homework). They were at the end of the slides for Tuesday's class.

A repeating decimal

Write

$$0.230769\,230769\,230769\cdots$$

as a fraction. (What repeats is the sequence 230769 of length 6.)

This number is

$$\frac{230769}{10^6}+\frac{230769}{10^{12}}+\frac{230769}{10^{18}}+\cdots.$$

That's a geometric series.

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That's a geometric series.

The formula for the sum of a geometric series is

$$a + ar + ar^2 + \cdots = \frac{a}{1-r}.$$

Here, $a = \dfrac{230769}{10^6}$ and $r = \dfrac{1}{10^6}$, so the sum is $\dfrac{230769}{999999}$.

It is possible to simplify this fraction: it's actually $\dfrac{3}{13}$.
(Simplifying would not be required on an exam.)

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A slide from Tuesday's class

This slide was about the ratio test:

Take an infinite series $\sum_{n=1}^{\infty} a_n$ with the a_n non-zero (except perhaps for the first few). The a_n are allowed to be negative. Suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists. Then

- If the limit is less than 1, the series converges.
- If the limit is greater than 1, the series diverges (and in fact its nth term doesn't approach 0).

If the limit is 1, the test says nothing about convergence.

Before giving examples where the ratio test applies, I want to make some statements that are intend to orient you in the world of infinite series.

If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

does not exist, the ratio test says nothing.

If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1,$$

the ratio test says nothing.

Two examples

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. For this series,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1.$$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. For this series,

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If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1,$$

the terms a_n are getting bigger in size and the nth term is not approaching 0. The series diverges.

Example: $\sum_{n=1}^{\infty}\frac{n^n}{n!}$, which we saw on Tuesday. The limit

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

is infinite, which in this context we consider to be > 1.

If you are trying to read these slides at home to review what we did in class, please remember that the slides need to be supplemented by what was written on paper (document camera) and on the board.

In case

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1,$$

the ratio test states that $\sum a_n$ is a convergent series. Through the next slides, we will try to understand why that is.

Positive series either converge or “sum to ∞”

The series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, as I said on Tuesday. (The sum is $\frac{\pi^2}{6}$, which is further amusing information.)

The similar-looking sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges by going off to infinity at a snail’s pace.

The key fact is that a positive series doesn’t have much choice—it’s either like the first series (convergence!) or the second series. A sum of positive numbers either exists as a definite number or “goes to infinity” (maybe very slowly).

Comparison

For positive series, being smaller than a convergent series is enough to imply convergence: if a bigger series doesn't go off to infinity, then a smaller series can't go off to infinity either.

Example: because $\sum \frac{1}{2^n}$ converges, the series

$$\sum_{n=1}^{\infty} (\sin^2 n) \frac{1}{2^n}$$

converges as well.

The other side of the coin is that being bigger than a divergent series guarantees divergence. For example,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

diverges because

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

For series that are not necessarily positive, divergence does *not* mean “going off to infinity.” An example to keep in mind is the series

$$1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots .$$

The partial sums are 1, 0, 1, 0, 1, 0,. . . . This sequence of partial sums does not converge to anything, so the series does not converge. It diverges.

Absolute convergence implies convergence

The next thing to know is that a series $\sum a_n$ converges if it converges "absolutely" in the sense that the sum of the absolute values of the a_n converges:

$$\sum |a_n| < \infty \Longrightarrow \sum a_n \text{ convergent.}$$

For example,

$$1 + \frac{1}{2} + \frac{1}{4} - \frac{1}{16} - \frac{1}{32} - \frac{1}{64} + \cdots$$

(three plusses, three minuses, three plusses, three minuses,...) converges because the sequence of absolute values

$$\sum \frac{1}{2^n}$$

converges.

The ratio test concerns ratios of absolute values $\left|\dfrac{a_{n+1}}{a_n}\right|$. If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1,$$

the ratio test shows that $\displaystyle\sum_{n=1}^{\infty}|a_n|$ converges, which is the statement that $\sum a_n$ converges absolutely.

So the ratio test is really about sums of positive numbers.

Because absolute convergence implies convergence, it follows that $\sum a_n$ converges.

End of explanation of ratio test

Suppose that we have a series of positive numbers $\sum a_n$ and that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1.$$

Then, roughly speaking, each term a_{n+1} is close to $L \cdot a_n$. In words, each term is around L times the previous term. This gives the idea that $a_n \approx L^{n-1} a_1$ for all n, so that the sum $\sum a_n$ is like the geometric series

$$a_1 + La_1 + L^2 a_1 + L^3 a_1 + \cdots ,$$

which converges. Hence the series $\sum a_n$ is convergent.

Ratio test examples

$$\sum_{n=1}^{\infty} \frac{c^n}{n!},$$

where c is a fixed number. If $a_n = \dfrac{c^n}{n!}$, then $\dfrac{a_{n+1}}{a_n} = c\dfrac{1}{n+1}$ and

$$\left|\frac{a_{n+1}}{a_n}\right| = |c|\frac{1}{n+1} \to 0.$$

Thus the series converges. Some of you may have learned that the sum is e^c.

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$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

We already know that the series diverges, but what happens if we apply the ratio test? Well,

$$a_{n+1}/a_n = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{1}{n+1} = \left(1 + \frac{1}{n}\right)^n \to e > 1.$$

Thus the series diverges.

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

The fraction a_{n+1}/a_n now approaches $1/e$ instead of e. Since $\frac{1}{e} < 1$, the series converges.

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{36} \cdots ,$$

where each term is alternately $1/2$ or $1/3$ times the previous term. The sequence

$$\frac{a_{n+1}}{a_n}$$

looks like

$$1/2, 1/3, 1/2, 1/3, 1/2, \cdots$$

and doesn't approach a limit. Hence the ratio test gives no information.

Actually, the series converges by comparison with

$$1/2 + 1/4 + 1/8 + \cdots .$$