# Derivatives and series FTW!

Math 10A

September 19, 2017

# Mehek

Mehek Mohan came to visit last week. Please contact me if you'd like me to introduce you to her (and vice versa).

The fifth Math 10A breakfast was on Friday...

. . . and the sixth on Monday:


The next breakfast is tomorrow morning.

A new breakfast has just been created for Wednesday, October 4. Sent email to sign up for that date!

Pop-in lunch on Friday at noon.

# Note from a former student

Received this morning:

> *I graduated last year and now I have just started my graduate education over the other side of the bay (no hint needed for the name). I saw your Facebook post for the breakfast that you just hosted and it suddenly reminded me of the days back when our class used to have breakfast with you. I just wanted to say thank you for teaching us and caring so much about your students!*

# Announcements

- Yesterday's office hour was cancelled because of a committee meeting.
- Special office hour *today* at 11AM in 885 Evans.
- Regular SLC office hour on Wednesday at 10:30AM.
- The midterm exam on Tuesday, September 26 "covers" everything in the course discussed through today.

# A good l'Hôpital's rule example

Find

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}.$$

This is a good problem, but it's done out on the next slides. I propose *not* to discuss it in lecture today.

# A good l'Hôpital's rule example

We are in the world of l'Hôpital because numerator and denominator are both approaching 0. The Rule allows you to replace numerator and denominator by their respective derivatives; you get

$$\lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x} = -1/6.$$

To get from the first of these limits to the second, I applied l'Hôpital a second time. Since $\dfrac{\sin x}{x}$ is known to have limit 1, I just wrote down the answer $-1/6$.

Alternatively, I could have applied l'Hôpital a third time.

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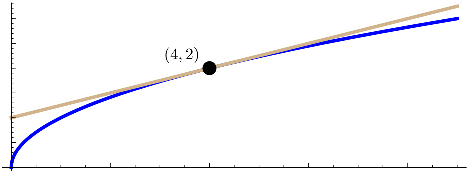
# Linear approximation

Suppose $f$ has a derivative at $c$ and consider the curve $y = f(x)$ and the line tangent to the curve at $(c, f(c))$. For $x$ near $c$, the line does not stray far from the curve.




On this graph, $f(x) = \sqrt{x}$, $a = 4$, $f(a) = 2$. The tangent line has equation $y = 1 + \frac{x}{4}$. We might be willing to accept the approximation

$$f(4.1) \approx 1 + \frac{4.1}{4} = 2.025.$$

That's linear approximation in a nutshell. A typical question:

- You lost your calculator and want to compute by hand a reasonable approximation to $\sqrt{4.1}$ by using the tangent line. What answer would you get? [2.025]
- You found your calculator! What is the decimal value of $\sqrt{4.1}$? [2.02484567313166]

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In general, the equation of the line tangent to $y = f(x)$ at the point $(a, f(a))$ has slope $f'(a)$ and equation

$$y = f(a) + f'(a)(x - a)$$

as you see from the point–slope formula for a line:

$$\frac{y - f(a)}{x - a} = f'(a).$$

# Higher approximations

If you know $f''(a)$ as well as $f'(a)$, you can refine the approximation

$$y \approx f(a) + f'(a)(x - a)$$

by adding on a quadratic term:

$$y \approx f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2}.$$

For example, taking $a = 0$ and $f(x) = e^x$, we get the approximation

$$e^x \approx 1 + x + x^2/2$$

for $x$ near 0.

For more examples, you might want to look at my slides from last year's Math 10A when I discussed this cluster of topics.

I looked at these slides last week and thought that they had aged well.

# Series

The main reference for infinite series is the .pdf file on differential calculus that's available on `bCourses`.

Here are some topics that we want to know solidly:

- To each series we associate the sequence of $n$th terms of the series and also the sequence of partial sums of the series.
- To say a series converges (to $S$) is to say that the sequence of partial sums converges (to $S$).
- If a series converges, its $n$th term approaches 0.
- There are divergent series whose $n$th terms approach 0.
- Geometric series are easy in the sense that we know whether or not they converge and we know their sums when they converge.

# Ratio test

We also want to know this test for convergence:
Take an infinite series $\sum_{n=1\infty} a_n$ with the $a_n$ non-zero (except perhaps for the first few). The $a_n$ are allowed to be negative. Suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

exists. Then

- If the limit is less than 1, the series converges.
- If the limit is greater than 1, the series diverges (and in fact its $n$th term doesn't approach 0).

If the limit is 1, the test says nothing about convergence.

# Question from 2016

Does this series converge: $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$?

This is $\displaystyle\sum_{n=1}^{\infty} a_n$, with $a_n = \dfrac{1}{n}-\dfrac{1}{n+1}$.

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What are the partial sums? They are: $(1 - \frac{1}{2})$,

$(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) = 1 - \frac{1}{3}$,

$(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) = 1 - \frac{1}{4}$, etc.

The $n$th partial sum is $s_n = \left(1 - \frac{1}{n+1}\right)$. Since $s_n \to 1$ as $n \to \infty$, the series *converges* and its sum is 1.

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# Factorials

Notation: $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$. Thus:

$$
\begin{aligned}
0! &= 1 \text{ (that's a convention)}, \\
1! &= 1, \\
2! &= 2, \\
3! &= 6, \\
4! &= 24, \\
5! &= 120, \\
&\cdots \\
10! &= 3628800, \\
&\cdots \\
20! &= 2432902008176640000,
\end{aligned}
$$

etc.

What can one say about $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$?

The $n$th term is bigger than 1, so the $n$th terms do not approach 0. Therefore the series diverges.

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# A repeating decimal

Write

$$0.230769\,230769\,230769\,\cdots$$

as a fraction. (What repeats is the sequence 230769 of length 6.)

This number is

$$\frac{230769}{10^6} + \frac{230769}{10^{12}} + \frac{230769}{10^{18}} + \cdots.$$

That's a geometric series.

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The formula for the sum of a geometric series is

$$a + ar + ar^2 + \cdots = \frac{a}{1 - r}.$$

Here, $a = \dfrac{230769}{10^6}$ and $r = \dfrac{1}{10^6}$, so the sum is $\dfrac{230769}{999999}$.

It is possible to simplify this fraction: it's actually $\dfrac{3}{13}$.
(Simplifying would not be required on an exam.)

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