Approximations, Limits

Math 10A

September 14, 2017

Something from the Document Camera

While writing in pen, I mentioned this: Euler conjectured in the 18th century that a perfect fourth power cannot be the sum of three perfect fourth powers. The first non-zero solution to $a^4 + b^4 + c^4 = d^4$ was uncovered in 1988 when Noam Elkies of Harvard found:

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4.$$

As I said, the search that he did on his laptop was preceded by a lot of intelligent calculation.

Typical SLC office hour photo

Announcements

My next office hour will be on Monday at 1:30PM in 885 Evans Hall.

Faculty Club pop-in lunch tomorrow at high noon.

The breakfast on Monday, September 25 at 8AM still has four places available. Send email if you'd like one.

The midterm exam on Tuesday, September 26 “covers” everything in the course discussed through Tuesday, September 19.

The Mean Value Theorem

Suppose that $f(x)$ is a differentiable function on an interval and that a and b are points inside the interval with $a < b$. The *average* rate of change of f on the interval $[a, b]$ is

$$\frac{f(b) - f(a)}{b - a}.$$

The *instantaneous* rate of change of f at a point c is $f'(c)$.

The Mean Value Theorem (MVT) states that there's a c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This means that some tangent line between a and b is parallel to the secant line connecting $(a, f(a))$ and $(b, f(b))$.

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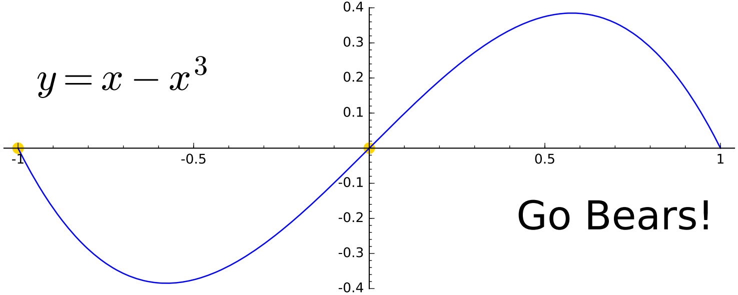
Random Example

Take $y = x - x^3$, $a = -1$, $b = 0$. Then $f(a) = f(b) = 0$, and the MVT asserts that there's a c between -1 and 0 where the tangent line is horizontal.



If $f(a) = f(b) = 0$, the MVT is also known as Rolle's Theorem.

In the picture, it looks to me as if c is around -0.55.

What is the actual value of c?

It's a point where the derivative of $x - x^3$ is 0. The derivative is $1 - 3x^2$, so $x = -1/\sqrt{3} \approx -0.577$.

The derivative is 0 also at $+1/\sqrt{3}$, but we were looking for a c between -1 and 0.

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When I first typed these slides, I somehow typed “Meal Value Theorem” for “Mean Value Theorem.” So you see what’s on my mind.

Intermediate Value Theorem

The Intermediate Value Theorem concerns a continuous function on an interval $[a, b]$. (Suppose again that a is less than b.) It states that every number between $f(a)$ and $f(b)$ is a value of f on the interval.

For example, if $f(a)$ is negative and $f(b)$ is positive (or vice versa), then 0 is a value of f on the interval. In plain language, this means that there is a c between a and b such that $f(c) = 0$. In plainer language: f has a root between a and b.

Intermediate Value Theorem example

Take $f(x) = x^3 - x - 1$. Then $f(1) = -1$ and $f(2) = 5$. It follows that there is a solution to $x^3 - x - 1 = 0$ between $x = 1$ and $x = 2$.

"Using technology," as Schreiber likes to say, we obtain the numerical approximation 1.32471795724746 for the root.

My technology is SageMath, also known as "Sage." Sage can make graphs with "Go Bears!" Can your calculator do that?

We will calculate some limits

$$\lim_{x \to 0+} x^{1/x}.$$

What is the value of this limit?

It’s 0. If you take a small number and raise it to a huge power, the result is minuscule.

This is not problematic or challenging if you reason calmly.

$$\lim_{x \to \infty} x^{1/x}.$$

What is the value of this limit?

This is a *tug of war* situation because the exponent $1/x$ is going to 0 and trying to make the expression $x^{1/x}$ look like 1. At the same time, the base x is going to infinity. Formally, we have 1^{∞}, which is one of several *indeterminate forms*.

The best way to untangle an exponential expression is to take the log:

$$\ln\left(x^{1/x}\right) = \frac{\ln x}{x}.$$

Heuristically, you might say that the numerator, $\ln x$ is “a lot smaller” than x and thus the fraction ought to approach 0.

But how to justify your heuristic idea?

l'Hôpital's Rule

l'Hôpital's Rule states that

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

in situations that look formally like $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

You can replace the numerator and denominator by their derivatives.

l'Hôpital's Rule applies to $\lim_{x\to\infty} \frac{\ln x}{x}$:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Since $\lim_{x\to\infty} \frac{\ln x}{x} = 0$,

$$\begin{aligned} \lim_{x\to\infty} x^{1/x} &= \lim_{x\to\infty} e^{\ln(x^{1/x})} \\ &= e^{\lim_{x\to\infty} \ln(x^{1/x})} = e^0 = 1. \end{aligned}$$

We used that $\ln\left(x^{1/x}\right) = \frac{\ln x}{x}$.

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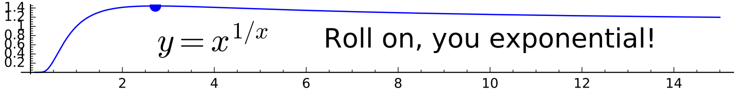
Graphing $y = x^{1/x}$



"Using technology," we can just graph the thing.

The blue half-moon is located at the point on the curve where the function takes its maximum.

What point is that?

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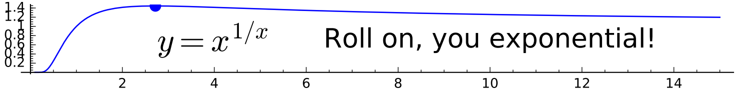

"Using technology," we can just graph the thing.

The blue half-moon is located at the point on the curve where the function takes its maximum.

What point is that?

We can use the fact that the derivative of a function vanishes when the function has a (local) maximum or minimum. (The proof of this fact will be revealed on the chalkboard.)

Are we willing to differentiate $x^{1/x}$?

This function is $e^{f(x)}$, $f(x) = \dfrac{\ln x}{x}$. Its derivative is then $e^{f(x)}f'(x) = x^{1/x}f'(x)$. Hence the derivative of $x^{1/x}$ is 0 at x if and only if $f'(x) = 0$.

This is not surprising because $x^{1/x}$ has a maximum when $f(x)$ has a maximum.

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This is not surprising because $x^{1/x}$ has a maximum when $f(x)$ has a maximum.

Now $f(x) = \frac{\ln x}{x}$, so $f'(x)$ is a fraction with denominator x^2 and numerator

$$d(x)n'(x) - n(x)d'(x),$$

where $n(x) = \ln x$ and $d(x) = x$. The numerator is then $x\frac{1}{x} - \ln x \cdot 1 = 1 - \ln x$. The fraction is 0 when $\ln x = 1$, i.e., when $x = e$.

Conclusion: the coordinates of the half-moon are $(e, e^{1/e})$.

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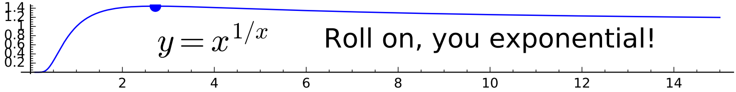
A challenge



The graph of $y = x^{1/x}$ is concave down until x is around 6; then it's concave up.

Said differently: the derivative is decreasing until we get to some *point of inflection*, which occurs around $x = 6$. Then the derivative increases slightly (i.e., gets less negative) to the right of the inflection point, all the way out to infinity.

What is the x-coordinate of the point of inflection?

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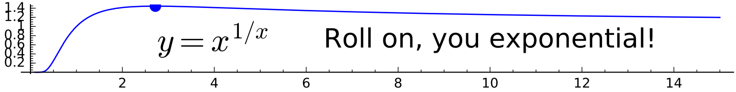

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What is the x-coordinate of the point of inflection?

To find the x-coordinate of the inflection point, you need to find the value of x at which the *second derivative* of $x^{1/x}$ is 0.

We will calculate some limits

What's the value of this limit?

$$\lim_{x\to\infty}(\sqrt{x^2+5x+7}-x)$$

Formally, both $\sqrt{x^2+5x+7}$ and x are approaching ∞, so we have

$$\infty - \infty,$$

which is another tug of war. It's another *indeterminate form*.

As many of you know, a good strategy here is to write

$$\begin{aligned}\sqrt{x^2+5x+7}-x &= (\sqrt{x^2+5x+7}-x)\cdot\frac{\sqrt{x^2+5x+7}+x}{\sqrt{x^2+5x+7}+x}\\ &= \frac{5x+7}{\sqrt{x^2+5x+7}+x}\\ &= \frac{5+7/x}{\sqrt{1+5/x+7/x^2}+1}.\end{aligned}$$

As $x\to\infty$, the fraction approaches $\dfrac{5}{1+1}=5/2$.

We will calculate some limits

What's the value of this limit?

$$\lim_{x \to -\infty} (\sqrt{x^2 - 5x + 7} + x)$$

It's the same limit as on the last slides.

The strategy is to let $t = -x$, so $t \to +\infty$, and

$$\lim_{x \to -\infty} (\sqrt{x^2 - 5x + 7} + x) = \lim_{t \to \infty} (\sqrt{t^2 + 5t + 7} - t).$$

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Find

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}.$$

We will calculate some limits

We are in the world of l'Hôpital because numerator and denominator are both approaching 0. The Rule allows you to replace numerator and denominator by their respective derivatives; you get

$$\lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \lim_{x\to 0} \frac{-\sin x}{6x} = -1/6.$$

To get from the first of these limits to the second, I applied l'Hôpital a second time. Since $\frac{\sin x}{x}$ is known to have limit 1, I just wrote down the answer $-1/6$.

Instead I could have applied l'Hôpital a third time.

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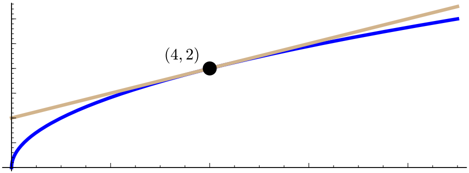
Linear approximation

Suppose f has a derivative at c and consider the curve $y = f(x)$ and the line tangent to the curve at $(c, f(c))$. For x near c, the line does not stray far from the curve.



On this graph, $f(x) = \sqrt{x}$, $a = 4$, $f(a) = 2$. The tangent line has equation $y = 1 + \frac{x}{4}$. We might be willing to accept the approximation

$$f(4.1) \approx 1 + \frac{4.1}{4} = 2.025.$$

That's linear approximation in a nutshell. A typical question:

- You lost your calculator and want to compute by hand a reasonable approximation to $\sqrt{4.1}$ by using the tangent line. What answer would you get? [2.025]
- You found your calculator! What is the decimal value of $\sqrt{4.1}$? [2.02484567313166]

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For more examples, you might want to look at my slides from last year's Math 10A when I discussed this cluster of topics.

I looked at these slides last night and thought that they were really good.