Differential Equations

Math 10A

October 19, 2017

The photo of me on the title page was taken in Taipei on August 1. I put it up on Instagram (`realkenribet`) on Tuesday.

Midterm #2

The second midterm will be one week from today, 8:10–9:30AM in 155 Dwinelle.

It is cumulative in principle, but it will stress the second third of the course.

It "covers" everything through today's course meeting.

Questions? Post them on piazza—that seems best.

Breakfast, anyone?

New breakfast: Wednesday, October 25 at 9AM. Please send me email to sign up. (This continues the tradition of breakfasts the day before our exams.)

I'd be down for November 1 or November 3 as well. Time to be determined. Neither is scheduled for now—I await your requests.

Lunch and Dinner

Pop-in lunch tomorrow (Friday) at noon. Meet in the Great Hall of the Faculty Club.

Foothill DC dinner, Friday, Nov. 3, 6:30PM.

Slides for this course

Last Thursday, our class was visited by Richard Freishtat, the director of the UC Berkeley Center for Teaching and Learning. I asked him to come to give me feedback on my teaching.

I was pleased to learn that he thinks I'm doing a good job. Hey, I was like blushing.

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But he did write this in his notes:

> *never reads/repeats slide text, but truly complements and expands on it. There is a unique value in coming to Ribet's class, to get his explanation that cannot be found anywhere else.*

The takeaway: when you need to miss an occasional class, you won't be able to pick up on everything that took place during the class meeting just by looking at slides.

On `bCourses`, don't forget that there's a course capture for the class and that the document camera pages are available as scans.

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Today

We will study *differential equations*, equations involving a function and its derivative(s). The aim is to determine the function.

For example: *Find* $f(x)$ *given that* $f'(x) = \cos x$ *and* $f(0) = 1$*.*

Here, we recognize an antiderivative problem and see that

$$f(x) = \sin x + C.$$

Setting $x = 0$, we find that $C = 1$ and emerge with the formula

$$f(x) = 1 + \sin x.$$

Important principle

If $F' = 0$, then F is constant.

Why is this true?

Heuristic reason: if a function has derivative identically equal to 0, it's not changing. It's a constant.

Sophisticated reason: this follows from the mean value theorem. Namely, if a and b are two different numbers, then

$$F(b) - F(a) = F'(c)(b - a) \text{ for some } c \text{ between } a \text{ and } b.$$

Since $F'(c) = 0$, $F(b) = F(a)$. To say that all values of a function are the same is to say the function is constant.

Important principle

If two functions have the same derivative, their difference is a constant.

Why? Apply previous principle to the difference between the two functions. The derivative of the difference is 0.

Today

Two distinct goals:

- To recognize a differential equation from a "real-life" scenario. (The mathematician's concept of "real life" may not be yours.) A key word is "modeling."
- To solve a differential equation symbolically.

I will begin with modeling.

Exponential growth and the logistic equation

A scenario is presented in which the derivative of a function of time is a constant times the function:

$$\frac{dN}{dt} = RN,$$

where R is a constant. For example, N might be the number of bacteria in some colony. . . .

This is a DE (differential equation) and the general solution is

$$N(t) = Ce^{Rt},$$

where C is a constant. If we set $t = 0$, we see that

$$C = N(0).$$

When the "initial value" $N(0)$ is furnished, and R is known, then $N(t)$ is completely determined.

Analysis

We can first check that $N(t) = Ce^{Rt}$ satisfies the equation. We just take the derivative of both sides of $N(t) = Ce^{Rt}$ and get

$$N'(t) = Ce^{Rt} \cdot R = RCe^{Rt} = RN(t).$$

We used the chain rule to differentiate e^{Rt}.

Conversely, how do we know that *every* solution to the equation may be written Ce^{Rt} for some constant?

Reformulation

If N is a solution, we want to know that $N/e^{Rt} = Ne^{-Rt}$ is a constant.

To see this, we apply the first *important principle*. To show something is a constant, it's enough to check that the derivative is 0. This is not hard (see doc camera).

Growth and decay

When R is positive, N is growing exponentially.

When R is negative, N is decaying exponentially.

When $R = 0$, N is constant (boring).

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Logistic stuff

In biology (a.k.a. real life), things don't continue to grow forever. There are brakes. Here is the book's notation:

If

$$\frac{dN}{dt} = RN \text{ where } R \text{ is a constant,}$$

we have exponential growth (or decay).

Replace the constant R by a product $r(1 - \frac{N}{K})$, where K is some limiting value of N (the most N could possibly be). The quantities r and K are constants; N continues to be a function of t.

Logistic stuff

$$\frac{dN}{dt} = r(1 - \frac{N}{K})N, \quad \text{where } r \text{ and } K \text{ are constants}$$

When N is small, the brake $(1 - \frac{N}{K})$ is near 1, so we're dealing essentially with

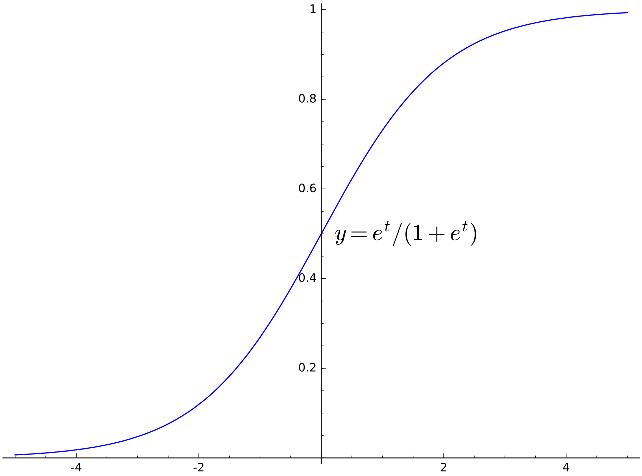
$$\frac{dN}{dt} = rN.$$

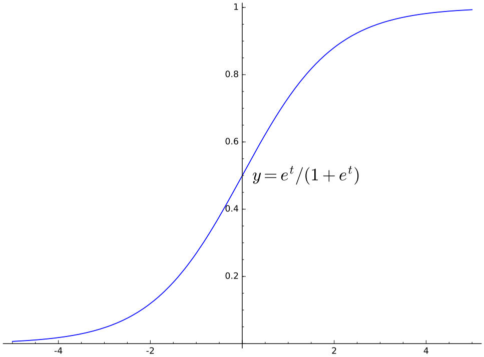
When N is very close to K, $(1 - \frac{N}{K})$ is close to 0, so the growth of N slows down.

What we expect is that N grows exponentially until it starts getting close to K and then its growth gets slower and slower.

A logistic function

Let $N(t) = \dfrac{e^t}{1+e^t} = \dfrac{1}{1+e^{-t}}$. You've seen this kind of function on HW.





For $t \to -\infty$, the function approaches $\frac{0}{1} = 0$. For $t \to +\infty$, the function approaches $\frac{1}{1+0} = 1$. At $t = 0$, the value of the function is $\frac{1}{2}$. You can see this behavior pretty well on the plot, which was made for t between -5 and $+5$.

Looking ahead

The domain of the logistic function N is the set of all real numbers. The function is defined everywhere! Meanwhile, the values of N are the real numbers between 0 and 1 (excluding both of them). Since probabilities are numbers between 0 and 1, the logistic function is important in probability theory.

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Logistic function, logistic equation

Check this out:

$$\frac{dN}{dt} = N(1 - N).$$

In other words, the *logistic function N* satisfies the logistic equation (with $r = K = 1$).

Note: sometimes mathematicians write $N(t)$ for the function, and sometimes just N. It's the same thing.

Here's the verification:

$$N(t) = \frac{e^t}{1+e^t},$$
$$1 - N(t) = \frac{(1+e^t) - e^t}{1+e^t} = \frac{1}{1+e^t},$$
$$N'(t) = \frac{(1+e^t)e^t - e^t e^t}{(1+e^t)^2} = \frac{e^t}{(1+e^t)^2}.$$

The right-hand function in the third equation is the product of the two functions above it. In other words,

$$N'(t) = N(t)(1 - N(t)).$$

Separable differential equations

A *separable* DE has the initial form

$$\frac{dy}{dt} = \text{ some messy function of } y \text{ and } t.$$

What makes it separable is that we can torment it algebraically until it has the form

$$\text{function of } y\, dy = \text{function of } t\, dt,$$

say

$$f(t)\, dt = g(y)\, dy.$$

We can then write

$$\int f(t)\, dt = \int g(y)\, dy$$

and solve the equation.

Separable differential equations

For example, consider the DE

$$\frac{dy}{dx} = 3xy, \quad \text{(§6.2, problem 23).}$$

(We have x instead of t.) This is separable:

$$\frac{1}{y}\,dy = 3x\,dx.$$

Integrating gives

$$\int \frac{1}{y}\,dy = \int 3x\,dx,$$

$$\ln y = \frac{3}{2}x^2 + C$$

$$y = \text{(some constant)}\, e^{\frac{3}{2}x^2}.$$

Check the answer

If $y = Ke^{3x^2/2}$, then $\frac{dy}{dx} = Ke^{3x^2/2} \cdot 3x = 3xy$.

One might ask: why not have $\ln(|y|)$ instead of $\ln y$ when we integrate $\frac{1}{y}\,dy$? If y is negative, then we're indeed dealing with $\ln(-y)$; we'd get

$$-y = \text{(some constant) } e^{\frac{3}{2}x^2}$$

and we could write instead

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