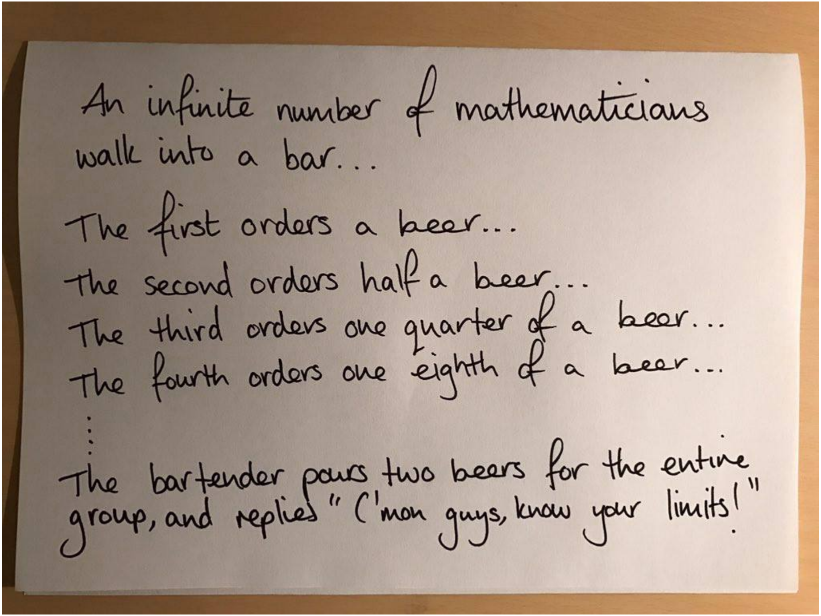

An infinite number of mathematicians walk into a bar...
The first orders a beer...
The second orders half a beer...
The third orders one quarter of a beer...
The fourth orders one eighth of a beer...
⋮
The bartender pours two beers for the entire group, and replies "C'mon guys, know your limits!"

Applications of Integration

Math 10A

October 17, 2017

Office Hour

My usual SLC office hour: 10:30AM–noon on Wednesdays.

Tomorrow: 10:00–11:30AM instead.

Because HW has been due on Tuesdays, I'm getting visits on Mondays (885 Evans) but not on Wednesdays. Stop by tomorrow to check in and say "hi."

Foothill Dinner

Friday, Nov. 3, 6:30PM.

Let's get up a big group!

Faculty Club “events”

The next pop-in lunch will be Friday, October 20 at High Noon. (Meet in the Faculty Club’s Great Hall as usual.)

Breakfast? I’d be down for October 23 or October 25 (next week) and for November 1 or November 3 as well. Nothing is scheduled for now—I await your requests.

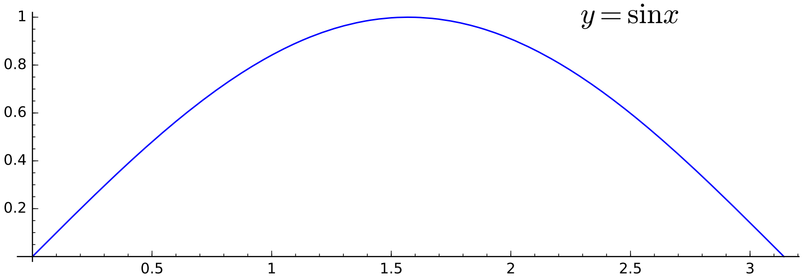
Today

The theme of today's class is that lots of frequently-encountered quantities are closet definite integrals.

Once we figure out that something is a definite integral, we can use numerical integration or the Fundamental Theorem of Calculus to evaluate the quantity.

Average value of a function

What is the average value of $\sin(x)$ on the interval $[0, 2\pi]$?



We answer the question by sampling lots of values of sin and taking the average of the values. Then we let the number of values $\longrightarrow \infty$.

Divide $[0, \pi]$ into n intervals of equal length:

$$[0, \frac{\pi}{n}], [\frac{\pi}{n}, \frac{2\pi}{n}], [\frac{2\pi}{n}, \frac{3\pi}{n}], \ldots, [\frac{(n-1)\pi}{n}, \pi].$$

Take points $x_1, x_2, \ldots, x_n$ with x_i in the ith interval. Take the average value of sin at those points:

$$\frac{1}{n}\left(\sin(x_1) + \sin(x_2) + \cdots + \sin(x_n)\right).$$

The *average value of* sin in the limit as $n \to \infty$ of this average.

Relation to Riemann sums

A Riemann sum in this setup would be

$$\frac{b-a}{n}\left(f(x_1)+\cdots+f(x_n)\right)$$

with $b=\pi$, $a=0$, $f=\sin$. Thus it would be

$$\frac{\pi}{n}\left(\sin(x_1)+\sin(x_2)+\cdots+\sin(x_n)\right).$$

As $n\to\infty$, this quantity approaches $\displaystyle\int_0^\pi \sin x\,dx$.

Conclusion: the average value of sin on $[0,\pi]$ is

$$\frac{1}{\pi}\int_0^\pi \sin x\,dx=\frac{2}{\pi}.$$

The average value of f on $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

for $a < b$.

Saving for retirement

Example 3 on page 419 of our textbook:

Starting at an early age, worker Peggy puts $2000 per year into a retirement account. The money is compounded continuously at the rate of 10% per year. How much will be in Peggy's account 40 years after she starts.

Remark: 10% is a totally unrealistic number at the present time.

Continuous compounding

Suppose I start a year with \$1 and my money is compounded continuously at the rate of 10% over the year. How much money do I have at the end of the year?

The answer is $e^{0.1} \approx \$1.10517$. In other words, I have about 10.5% more money than at the start of the year.

There are various ways to see this. One way is to view the answer as

$$\lim_{n\to\infty} \$ \left(1 + \frac{0.1}{n}\right)^n$$

and compute the limit.

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What about Peggy?

My reading of the problem was that Peggy puts in \$2000 at the start of the first year, \$2000 at the start of the second year,. . . , \$2000 at the start of the fortieth year. She examines her fortune at the end of the fortieth year.

The textbook seems to think that she is depositing her money continuously, not once per year. Realistically, she probably deposits money once per month or once per week.

Peggy's first \$2000 will be worth $\$2000(e^{0.1})^{40}$ at the end of the forty years. Her second \$2000 will be worth $\$2000(e^{0.1})^{39}$ at the end of the forty years. Her third \$2000 will be worth $\$2000(e^{0.1})^{38}$ at the end. . . . Her final \$2000 will be worth $\$2000(e^{0.1})$ at the end of the forty years.

Altogether, Peggy will have

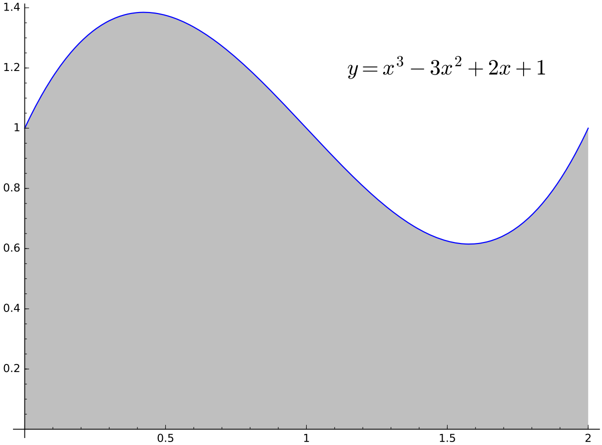
$$\$2000(\alpha + \alpha^2 + \alpha^3 + \cdots + \alpha^{40}),$$

where $\alpha = e^{0.1}$. This is a geometric series with sum

$$\$2000\alpha \frac{1 - \alpha^{40}}{1 - \alpha} \approx \$1,126,454.$$

Solids of revolution

Find the volume of the solid obtained by revolving about the x-axis the area under the curve $y = x^3 - 3x^2 + 2x + 1$ from $x = 0$ to $x = 2$.

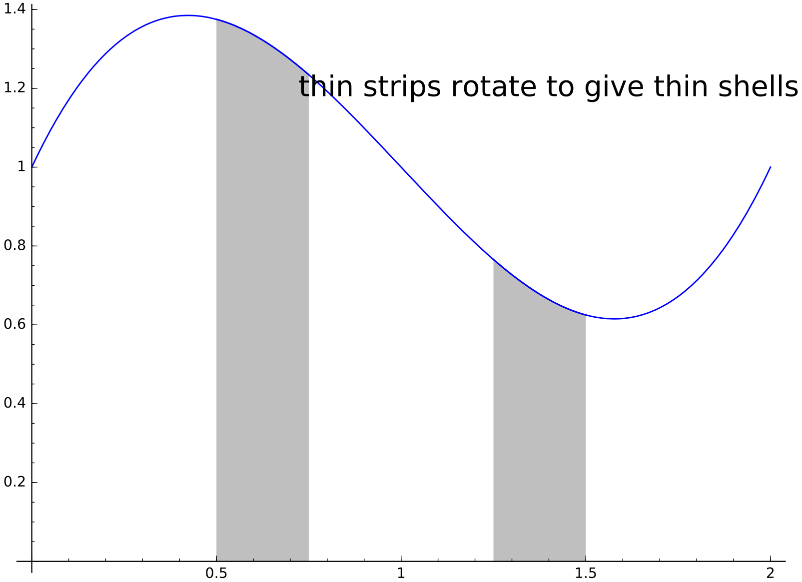


The principle (to be explained on doc camera) is that the shaded area can be divided up into thin strips. Each strip when revolved around the x-axis gives a solid that looks very much like a thin cylinder ("thin shell"). The volume of the solid is $\Delta x \cdot \pi f(x_i)^2$ (in general—here $f(x) = x^3 - 3x^2 + 2x + 1$). By taking more and more shells, which are becoming thinner and thinner, we get:

$$\text{Volume} = \pi \int_a^b f(x)^2 \, dx$$

as a general answer. Here

$$\text{Volume} = \pi \int_0^2 (x^3 - 3x^2 + 2x + 1)^2 \, dx.$$

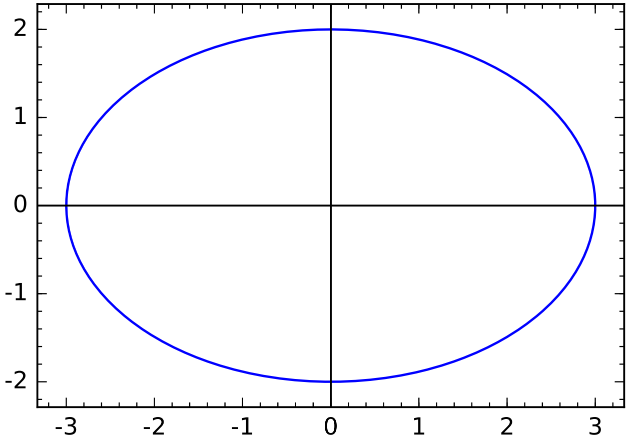

thin strips rotate to give thin shells
1.4
1.2
1
0.8
0.6
0.4
0.2
0.5
1
1.5
2

Numerically, our volume is

$$\pi \int_0^2 (x^6 - 6x^5 + 13x^4 - 10x^3 - 2x^2 + 4x + 1)\, dx = \frac{226\pi}{105}.$$

Disclaimer: I used “technology” to calculate the integral and see no great value in trying to do the calculation in class.

Calculate the volume of the football-shaped solid obtained by rotating the interior of the ellipse $\frac{y^2}{4} + \frac{x^2}{9} = 1$ about the x-axis.



This problem might have tripped up some 2016 students because the bottom part of the interior is irrelevant. We are thus talking about rotating the area *under the curve* $y = 2\sqrt{1 - \frac{x^2}{9}}$ about the x-axis.

The general idea is to consider a non-negative function $y = f(x)$ and to rotate the area under the curve from $x = a$ to $x = b$. Here we just have another special case.

The answer is going to be

$$\pi \int_{-3}^{3} 4(1 - \frac{x^2}{9})\, dx.$$

This works out to be 16π; I leave the calculation to you.

The volume of a sphere of radius r is known to be $\frac{4}{3}\pi r^3$. We could figure that out by viewing a sphere as a solid of revolution. Here the answer is somehow $\frac{4}{3}\pi(2^2 \times 3) = 16\pi$.

We could do the general figure $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and get all these numbers in one fell swoop.

2016 Final Exam

Determine the volume of the solid obtained by revolving the area under the curve $y = x^2 + 1$ from $x = 0$ to $x = 2$ about the x-axis.

Since we've already done two problems, this one is mainly for reference. I invite you to work out the answer.

2016 Final Exam

For n a positive integer, evaluate $\int_1^n \ln x \, dx$.

We calculated $\int \ln x \, dx$ last week and found $x \ln x - x + C$ as the antiderivative. Knowing this, you can calculate the definite integral as

$$(x \ln x - x) \Big]_1^n = n \ln n - n + 1.$$

2016 Final Exam

Using trapezoidal approximation, derive the estimate

$$\int_1^n \ln x \, dx \approx \ln(n!) - \frac{1}{2} \ln n.$$

(Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$.)

I won't do this in class but will remark that the trapezoidal approximation is the average of the left- and right-endpoint approximations.

How fast does n! grow?

Please take a look at this slide after class:

Because we have an estimate relating n! to an integral, and because we know how to evaluate the integral in question, we can get some information about the size of n!. See "How fast does it grow?" for more information.