Techniques of Integration: I

Math 10A

October 10, 2017

We had the tenth Math 10A breakfast yesterday morning:

If you'd like to propose a future event (breakfast, lunch, dinner), please let me know.

Techniques of Integration

To evaluate $\int f(x)\,dx$ (an antiderivative) or $\int_a^b f(x)\,dx$ (a number), we might try:

- Substitution = change of variables; we did some on Thursday;
- Integration by parts—the magic elixir;
- Numerical integration (for definite integrals);
- Integration via partial fractions (just a bit).

We will learn about these techniques this week—mostly through examples.

Substitution

On Thursday, we looked at

$$\int \cos(x^2)x\,dx.$$

Today:

$$\int e^{\cos x} \sin x \, dx.$$

We set $u = \cos x$, $\frac{du}{dx} = -\sin x$, $du = -\sin x \, dx$.

The integral becomes

$$-\int e^{u} \, du = -e^{u} + C = -e^{\cos x} + C.$$

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Find

$$\int_0^{\pi/2} e^{\cos x} \sin x \, dx.$$

We now know enough to write down the answer as

$$-e^{\cos x}\Big]_0^{\pi/2} = -e^{\cos(\pi/2)} + e^{\cos 0} = -e^0 + e^1 = e - 1.$$

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Alternative path

Find

$$\int_0^{\pi/2} e^{\cos x} \sin x \, dx.$$

Set $u = \cos x$ and $du = -\sin x \, dx$ as before. Now say:

> "As x ranges from 0 to $\pi/2$, $\cos x$ ranges from 1 to 0."

Accordingly:

$$\int_0^{\pi/2} e^{\cos x} \sin x \, dx = -\int_1^0 e^u \, du = -e^u \Big]_1^0 = e - 1.$$

At the end of the previous slide, we allowed ourselves to write $\int_1^0$ even though $1 < 0$.

Formally, $\int_b^a f(x)\,dx$ is defined to be $-\int_a^b f(x)\,dx$ for $a \leq b$.

The point of the previous example is that you never have to undo the substitution (i.e., go back from u to x) when computing a definite integral. Just write the limits of integration in terms of the new variable.

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A trig substitution

Recall this computation from the Time of Implicit Differentiation:

If $x = \tan y$, then

$$1 = (\sec^2 y)\frac{dy}{dx} = (1 + \tan^2 y)\frac{dy}{dx} = (1 + x^2)\frac{dy}{dx}$$

and thus

$$\frac{dy}{dx} = \frac{1}{1 + x^2}, \quad \frac{d(\tan^{-1} x)}{dx} = \frac{1}{1 + x^2}.$$

I've written "$\tan^{-1}$" for the inverse tangent, also called arctan.

Find an antiderivative of the function $\frac{1}{1+x^2}$.

In view of what we just did, we could come up with the answer $\tan^{-1} x$.

This is a bit unsatisfactory because we answered the question only be remembering a computation that we just did.

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The classic approach to computing $\int \frac{1}{1+x^2}\,dx$ is to introduce a "trigonometric substitution" out of the blue:

Let $x = \tan u$, $\frac{dx}{du} = \sec^2 u$, $dx = \sec^2 u\,du$. Then

$$\int \frac{1}{1+x^2}\,dx = \int \frac{1}{1+\tan^2 u}\sec^2 u\,du = \int 1\,du = u + C.$$

We finish by saying that $u = \tan^{-1} x$, so the answer is $\tan^{-1} x + C$.

Find $\int_0^\infty \frac{1}{1+x^2}\, dx$.

By definition, this is $\lim_{b\to\infty} \int_0^b \frac{1}{1+x^2}\, dx$.

We computed the indefinite integral: $\tan^{-1} x$. Since $\tan^{-1}(0) = 0$, $\frac{1}{1+x^2}\, dx = \tan^{-1} b$ and

$$\int_0^\infty \frac{1}{1+x^2}\, dx = \lim_{b\to\infty} \tan^{-1} b = \frac{\pi}{2}.$$

The area under the entire graph of $y = \frac{1}{1+x^2}$ is $2 \cdot \frac{\pi}{2} = \pi$ by symmetry.

BREAK

By Parts

The technique of *integration by parts* attempts to reverse-engineer the product formula

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx},$$

where u and v are functions of x. Symbolically,

$$d(uv) = udv + vdu, \quad \int d(uv) = \int u\,dv + \int v\,du.$$

In other words,

$$uv = \int u\,dv + \int v\,du, \quad \int u\,dv = uv - \int v\,du.$$

We are not bothering to write "$+C$" in these formulas because indefinite integrals include a tacit $+C$ in their values.

Say that again

$$\int u\,dv = uv - \int v\,du,$$

when u and v are functions of x. When we are confronted with an integral, it's up to us to decide how to use the formula: we introduce u and v so that $\int u\,dv$ is the integral we are handed and so that $\int v\,du$ seems to be more appealing.

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Suppose someone wants an antiderivative of $x \sin x$:

$$\int x \sin x \, dx = ??$$

We introduce $u = x$ and write tentatively $dv = \sin x \, dx$ in order to see whether we can find a v so that $dv = \sin x \, dx$. This equation amounts to

$$\frac{dv}{dx} = \sin x \Longrightarrow v = -\cos x.$$

We could have taken $v = -\cos x + 31$ but the choice of v is ours, so we pick the simple $v = -\cos x$.

Now

$$\int x \sin x \, dx = \int u \, dv \text{ with } u = x, v = -\cos x.$$

The formula

$$\int u \, dv = uv - \int v \, du$$

becomes

$$\int x \sin x \, dx = -x \cos x + \int \cos x \, dx.$$

The right-hand side is

$$-x \cos x + \sin x + C,$$

so we have learned that

$$\int x \sin x \, dx = -x \cos x + \sin x + C.$$

As a check, let's compute the derivative of $-x \cos x + \sin x$:

$$[-x \cdot (-\sin x) - 1 \cdot \cos x] + \cos x = x \sin x,$$

as we hoped.

Another example

Compute $\int_0^\infty xe^{-x}\,dx$.

Do it first without the factor x:

$$\int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big]_0^b = \lim_{b\to\infty}(1-e^{-b}) = 1.$$

Next, think about $\int xe^{-x}dx$ in terms of integration by parts.

Again we take $u = x$, and this time we take $v = -e^{-x}$ to get $dv = e^{-x}du$.

The formula $\int u\,dv = uv - \int v\,du$ gives

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx,$$

so

$$\int_0^b xe^{-x}\,dx = -xe^{-x}\Big]_0^b + \int_0^b e^{-x}\,dx = (0 - be^{-b}) + \int_0^b e^{-x}\,dx.$$

Hence

$$\int_0^\infty xe^{-x}\,dx = \int_0^\infty e^{-x}\,dx - \lim_{b\to\infty} be^{-b} = 1 - 0 = 1.$$

The value $\lim_{b\to\infty} b/e^b = 0$ comes from l'Hôpital's Rule.

In this computation, we can streamline things as follows:

- we can carry the limits of integration along as we compute instead of doing the indefinite integral first;
- we can be real pros and symbolically take $b = \infty$ all along.

Confronted with $\int_0^\infty xe^{-x}\,dx$, we take $u = x$, $v = -e^{-x}$ as before and write directly

$$\int_0^\infty xe^{-x}\,dx = -xe^{-x}\Big]_0^\infty + \int_0^\infty e^{-x}\,dx = (0-0) + 1 = 1.$$

Maybe this is just packaging.