Derivatives: definition and computation

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Math 10A
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Announcements

The breakfasts tomorrow and Thursday are full, but there are spaces at the 8AM breakfast on September 13.

This is a breakfast from last semester. The gentleman facing the camera near the upper-left corner is the Chancellor.

Reminder: homework assignments are listed in the “homework” section of the class web page

https://math.berkeley.edu/~ribet/10A/.

Chapter 2 of the Schreiber book is available in .pdf form from bCourses.

Today

First, we want to make sure that we understand the definition of the derivative as a limit.

Next we want to see how the definition applies to compute derivatives of standard functions like the power functions and the exponential function.

Then we want to learn how to differentiate sums, products and quotients of functions that know how to differentiate.

Finally, we want to check that we can do examples.

Derivatives

If f is a function that is defined on an interval around a, then the derivative of f at a is the number

$$\lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

that represents the ("instantaneous") rate of change of f at a. If the limit exists, f is *differentiable* at a. If it doesn't exist, it doesn't exist.

Setting $b = a + h$, we can rewrite the definition:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Differentiability implies continuity

If f has a derivative at a, then it's *continuous* at a. This means:

$$\lim_{b \to a} f(b) = f(a).$$

That's because

$$f(b) - f(a) = \frac{f(b) - f(a)}{b - a} \cdot (b - a) \longrightarrow f'(a) \cdot 0 = 0$$

as $b \to a$.

Last week's example

If f is the squaring function ($f(x) = x^2$), then $f'(a) = 2a$ for all real numbers a.

On the street, one often hears: "$2x$ is the derivative of x^2."

Power functions

If f is a constant function (e.g., $f(x) = 42$ for all x), then $f'(a) = 0$ for all a.

If f is the identity function ($f(x) = x$ for all x), then $f'(a) = 1$ for all a.

If f is the inversion function ($f(x) = 1/x$ for all non-zero x), then $f'(a) = -1/a^2$ for all $a \neq 0$ (document camera).

Suppose that n is a positive integer and that f is the nth power function ($f(x) = x^n$). Then $f'(a) = na^{n-1}$ for all a (document camera). The formula works also for $n = -1$ and $n = 0$—look at the examples higher up on this slide. The book says (without justification) that it works for all real exponents n.

As noted last week and also a few minutes ago, people tend to think of the derivative as a function. For example, if $f(x) = x^n$, people say that the derivative of f is the function nx^{n-1}.

Then there's this notation: the derivative $f'(x)$ is often written

$$\frac{d}{dx}f(x).$$

The idea is that $\frac{d}{dx}$ is an "operator" that means *take the derivative!*

The exponential function

Suppose that $f(x) = e^x$. Then

$$f'(a) = \lim_{h \to 0} \frac{e^{a+h} - e^a}{h} = e^a \cdot \lim_{h \to 0} \frac{e^h - 1}{h} = e^a f'(0).$$

The whole point of e is that $f'(0) = 1$. In other words, the slope of the line tangent to $y = e^x$ at $(0, 1)$ is 1. (We said this last week.)

Hence $\frac{d}{dx}(e^x) = e^x$:

The exponential function is its own derivative.

By the way, people often write $\exp(x)$ for e^x.

If you've never studied any calculus before, you'll probably think that we're going pretty fast. I think that we are!

This stuff will sink in.

If you use a different base instead of e, there's a constant factor in front of the derivative:

$$\frac{d}{dx}(b^x) = (\ln b)b^x.$$

Here ln is the natural logarithm.

Indeed, the proof that we just gave shows that

$$\frac{d}{dx}(b^x) = C \cdot b^x,$$

where C is the derivative of b^x at $x = 0$:

$$C = \lim_{h \to 0} \frac{b^h - 1}{h}.$$

We need to show that $C = \ln b$. For this, note that $b = e^{\ln b}$, so $b^h = e^{h \ln b}$ and

$$\frac{b^h - 1}{h} = \frac{(e^{h \ln b}) - 1}{h} = \frac{(e^{h \ln b}) - 1}{h \ln b} \cdot (\ln b) \longrightarrow 1 \cdot \ln b.$$

Two easy computational rules

The derivative of a sum is the sum of the derivatives:

$$\frac{d}{dx}\,(f(x) + g(x)) = f'(x) + g'(x).$$

This is because "the limit of a sum is the sum of the limits" (see document camera).

The derivative of a constant times a function is that constant times the derivative of the function:

$$\frac{d}{dx}\,(cf(x)) = cf'(x).$$

Again, this is easy to see from the definition of the derivative as a limit (document camera).

Leibniz's formula

The formula in question is also known as the product rule:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x).$$

Heuristically, the formula can be explained by considering what happens if we have a product and both factors change a little bit. For example, suppose we take $48 = 12 \cdot 4$ and add 1 to each factor. We get

$$(12+1)(4+1) = 48 + 1 \cdot 4 + 12 \cdot 1 + 1 \cdot 1,$$

so that the change in the product is $1 \cdot 4 + 12 \cdot 1 + 1 \cdot 1$. The last term $1 \cdot 1$ is must smaller than the other two terms; most of the change comes from $1 \cdot 4$ and $12 \cdot 1$, which represent the change in the first factor times the second factor, plus the change in the second factor times the first factor.

It should not be painful to derive ("prove") the formula from the definition of the derivative as a limit. I'll do that on the document camera.

Quotient rule

Today's last formula is for the derivative of a quotient $\dfrac{n(x)}{d(x)}$:

$$\left(\frac{n}{d}\right)' = \frac{dn' - nd'}{d^2}.$$

"The denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, over the denominator squared."

The proof of this formula is similar in spirit to the proof of the product formula.

Examples

Differentiate

$$\frac{3x^2 - x + 17}{e^x}.$$

Answer from `Sage`:

$$-(3\,x^2 - x + 17)e^{-x} + (6\,x - 1)e^{-x}.$$

We can see this is right without writing much because the denominator of the derivative starts out as $(e^x)^2$, but both terms of the numerator have e^x as a factor.

Differentiate x^π with respect to x.

The answer is $\pi x^{\pi-1}$. As I mentioned earlier, the book tells its readers that the rule for differentiating x^n works for all real numbers $n \neq 0$: "At this point, we are equipped to prove the

power rule for any natural number n. Later, we shall prove the general power rule."
Actually, it It works for $n = 0$ as well, if you consider that x^0 is the constant function 1, whose derivative is 0.

Here we have $n = \pi$.

Find the derivative of

$$x^2 - \frac{1}{x^2} + \frac{5}{x^4}.$$

(This is exericise #18 on page 218 of the textbook.)