Approximation using derivatives
l'Hôpital's Rule

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Math 10A
September 13, 2016

We had two breakfasts last week. . .

and another this morning.

In addition, I am organizing a pop-in lunch tomorrow (Wednesday) at the Faculty Club at 12:15PM: People can show up at the lunch line, pay for themselves with cash or credit card ($10 minimum) and join our group. I usually hold lunches like this on Fridays at 12:30PM (and announce them on Facebook), but I will be away at the end of the week.

James is pinch-hitting

Because I am President-elect of the American Mathematical Society, I will need to make several short trips to the East Coast this semester.

The first such trip is at the end of this week. As a result:

- GSI James McIvor will take over Thursday's class here in VLSB.
- I won't be able to hold office hours on Thursday morning.

Please do come to my office hours, by the way. I want to meet you all. In particular, I'd love to get your feedback and suggestions. Also, you can ask me math questions!

The next breakfast

As announced on `piazza`, the next breakfast will be Wednesday, September 21 at 8:30AM. Some places are still left. To sign up, please send me email.

The next exam

Our first midterm is in nine days. It will be given right here (2050 VLSB) during the class period (3:40–5PM).

Some questions:

What does the exam cover? Answer: Everything that we studied and discussed through the class on September 15.

Are there proofs? Answer: No, but it's important for you to *explain* what you're doing as you write.

Can we bring in notes? Answer: Yes, everyone can bring in one two-sided page of notes ($8\frac{1}{2} \times 11$ inches). No devices (e.g., calculators) are allowed.

More questions and comments on `piazza`.

A math limerick

This was posted to my Facebook timeline. It's pretty well known:

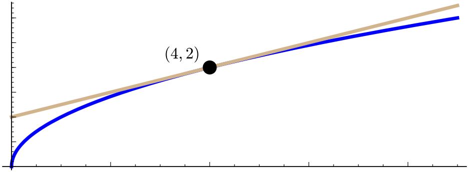
$$\frac{12 + 144 + 20 + 3\sqrt{4}}{7} + (5 \times 11) = 9^2 + 0.$$

The challenge is to narrate this equation in limerick form.

Spoiler at the end.

Linear approximation

If $f(x)$ is differentiable at a, then it is far from crazy to believe that $f(b)$ can be approximated by the value at b of the tangent line to $f(x)$ at $x = b$.



In the picture, $a = 4$, $f(a) = 2$ and the tangent line has equation $y = 1 + \frac{x}{4}$. We might be willing to accept the approximation

$$f(4.1) \approx 1 + \frac{4.1}{4} = 2.025.$$

Actually, $f(x) = \sqrt{x}$ in the picture. Since

$$\sqrt{4.1} = 2.0248\ldots,$$

the approximation is good enough to be used in practical calculations.

When I first started teaching calculus, the idea was that one could use approximations like this to compute numerical values of known functions (like log, sqrt, etc.). There were no handy “devices” that could compute the values in a flash.

Nowadays, we can use the same idea to approximate unmeasured values of functions in nature that are not typically given by exact formulas.

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What is the approximation, again?

In general, the tangent line has equation

$$\frac{y - f(a)}{x - a} = f'(a), \quad y = f(a) + f'(a)(x - a),$$

so that the approximation we'd get would be

$$f(b) \approx f(a) + f'(a)(b - a), \quad f(x) \approx f(a) + f'(a)(x - a).$$

Since algebraically

$$f(b) = f(a) + \frac{f(b) - f(a)}{b - a}(b - a),$$

our approximation results from some confidence that the fraction $\dfrac{f(b) - f(a)}{b - a}$ is close to $f'(a)$ when b is close to a.

Higher approximations

If f has derivative f', then the *second derivative* of f is the derivative f'' of f'. For example, if $f(x) = x^3$, then $f''(x) = 6x$. A refinement of the approximation $f(x) \approx f(a) + f'(a)(x - a)$ is the *second-order approximation*

$$f(x) \approx f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2.$$

There are third-order, fourth-order,..., nth order approximations if you want 'em. (Buzz phrase: Taylor polynomials.)

For example, take $f(x) = \sin x$, so that $f'(x) = \cos x$, $f''(x) = -\sin x$. Take $a = 0$. For x near 0, we have

$$\sin x \approx x,$$

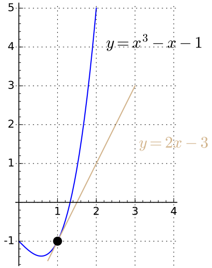
which is both the linear (first-order) and second-order approximation to sin; note that $f''(0) = 0$. The third-order approximation happens to be

$$\sin x \approx x - \frac{x^3}{6}.$$

Take $x = 0.1$. Then $\sin x$ is roughly 0.0998334. The linear approximation would be 0.1, which is not awful. The third-order approximation is $0.0998333\ldots$, which looks much better.

Newton's method

Newton's method leverages the idea of approximating a function by the associated tangent line. If we graph $y = x^3 - x - 1$, we see that there is a root between 1 and 2. The root seems to be closer to 1 than to 2.



We can start with $x = 1$ as an initial "guess" for the root. We then draw the tangent line to the curve at $x = 1$ and see where the line hits the x-axis. This gives us a revised guess for the location of the root.

The tangent line has equation $y = 2x - 3$; it hits the y-axis at $x = 1.5$, which is our revised estimate. (Looking at the graph, we might have started with 1.5 instead of 1 as our initial guess.)

In general, if a is an x-value near a root of $f(x)$, the revised estimate is given by the easy-to-remember formula

$$a - \frac{f(a)}{f'(a)}.$$

Indeed, the tangent line passes through $(a, f(a))$ and has slope $f'(a)$. It thus has equation

$$\frac{y - f(a)}{x - a} = f'(a).$$

When $y = 0$,

$$x - a = -\frac{f(a)}{f'(a)},$$

as claimed.

In the example $f(x) = x^3 - x - 1$, the formula

$$a - \frac{f(a)}{f'(a)}$$

becomes

$$\frac{2a^3 + 1}{3a^2 - 1}.$$

Starting with the guess $x = 1$, we obtain successive guesses

$$1.0, 1.5, 1.35, 1.3252, 1.32471817, 1.32471795724479, \ldots.$$

The root is approximately at $x = 1.32471795724475$, according to `Sage`.

Newton's method is a very powerful way to find roots!!

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Square root of 2

For $f(x) = x^2 - 2$, the roots of f are the square roots of 2. Using Newton's method, and starting with the initial guess 1, we get increasingly close estimates for $\sqrt{2} \approx 1.41421356237310$ by succesively replacing a guess a by

$$a - \frac{f(a)}{f'(a)} = \frac{1}{2}\left(a + \frac{2}{a}\right).$$

The successive guesses constitute a *sequence*. We met this sequence earlier in this course:

$$1, 1.5, 1.42, 1.4142, 1.41421356, \ldots.$$

Bisection

The bisection method is introduced in the book and occurs in at least one homework problem. The idea is to trap roots inside intervals whose length gets repeatedly halved.

For example, we look at the graph of $y = x^2 - 2$ and see that there's a root between 1 and 2. Is it between 1 and 1.5 or between 1.5 and 2? Evaluating $x^2 - 2$ at 1.5, we see that this function is positive at 1.5 but negative at 1. Therefore the root is between 1 and 1.5. The value of $x^2 - 2$ at 1.25 is negative, so the root is between 1.25 and 1.5.

This method is much slower than Newton's method!

l'Hôpital's Rule

The rule in question is a trick for evaluating limits that look formally like $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Briefly, the trick is to replace the functions in the numerator and denominator by their respective derivatives. If the limit can be evaluated after this replacement, the original limit exists and equals the limit of the modified fraction.

Example: $\lim_{x \to 0} \frac{\ln(1+x^2)}{x^2}$. This is not the simplest possible example, but it's a good one because it occurred in last week's homework.

As $x \to 0$, both numerator and denominator approach 0.

The l'Hôpital trick is to differentiate top and bottom:

$$\lim_{x \to 0} \frac{\ln(1+x^2)}{x^2} = \lim_{x \to 0} \frac{\frac{2x}{1+x^2}}{2x} =? \quad \text{OMG, it's 1!}$$

Why does this work? Half the time, it works because of the definition of the derivative. Consider the book's example $\lim_{x\to(\pi/2)^+} \dfrac{\cos x}{1-\sin x}$. The fraction can be rewritten

$$-\left(\frac{\cos x - \cos(\pi/2)}{x-\pi/2}\right)\Big/\left(\frac{\sin x - \sin(\pi/2)}{x-\pi/2}\right)$$

By definition, both expressions in $(\cdots)$ approach derivatives as $x \to \pi/2$. The fraction $\dfrac{\cos x - \cos(\pi/2)}{x-\pi/2}$ approaches $\cos'(\pi/2) = -1$. The fraction $\dfrac{\sin x - \sin(\pi/2)}{x-\pi/2}$ approaches $\sin'(\pi/2) = 0$. Taking into account the initial minus sign, we see that we're dealing formally with $\dfrac{1}{0}$, which is "infinite." Since x is bigger than $\pi/2$, $\dfrac{\sin x - \sin(\pi/2)}{x-\pi/2}$ is negative. Hence the limit is $-\infty$.

Spoiler

$$\frac{12 + 144 + 20 + 3\sqrt{4}}{7} + (5 \times 11) = 9^2 + 0$$

A dozen, a gross and a score
Plus three times the square root of four
Divided by seven
Plus five times eleven
Is nine squared and not a bit more.

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