Integration
Area under a curve

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Math 10A
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Announcements

I missed you guys!

I am back today but will be away for the rest of the week. I'm back again early next week (Monday–Wednesday) but will be away next Thursday and Friday.

When I leave next Thursday (October 13), I won't leave campus until morning. Accordingly, we can have breakfast!

The next breakfast will be on Thursday, October 13 at 9AM. (Note the late start, which was requested by the "seed participant.") You can sign up by sending me email. I hope to have a full house, just as we did for the first three breakfasts in September.

What are the key points?

If $f(x)$ is a reasonable function (e.g., continuous), and a and b are numbers, there is a number called $\int_a^b f(x)\,dx$. This is the *definite integral* of $f(x)$ over the interval $[a, b]$. The "dx" is often understood to mean "with respect to x." This will make more sense when f is a function of u and u is a function of x; for example, we could have $f(x) = \sin(x^2)$, so we'd say that $f(x) = \sin u$, where $u = x^2$.

When f is non-negative on $[a, b]$, $\int_a^b f(x)\,dx$ represents the area under $y = f(x)$ from a to b. (We can assume at first that a is less than b and figure out later what to do if $a > b$ or $a = b$.)

For example, $\int_{-1}^1 \sqrt{1 - x^2}\,dx = \frac{\pi}{2}$ because the area in question is a semicircle.

If f is not necessarily non-negative, $\int_a^b f(x)\,dx$ is a *signed area*. You count area above the x-axis as positive and area below the x-axis as negative. For example,

$$\int_{-\pi}^{\pi} \sin x\,dx = 0$$

because the area below the x-axis exactly cancels the area above the x-axis.

The number $\int_a^b f(x)\,dx$ is defined to be a limit of (signed) areas of rectangles as I'll show on the board or document camera. More generally, *area* in mathematics has to be defined. If you have a blob in the plane and want to figure out its area, you have to do something that involves a limit.

Notice for example that there are two possible definitions of the number π:

1. The area of a unit circle is π;
2. The circumference of a unit circle is 2π.

To see that these definitions give you the same number π, you have to relate the area and the perimeter of a circle. Archemedes did this a long time ago. The idea is that a circle is the "limiting case" of a regular polygon with a large number of sides. The area of the polygon can be decomposed as the sum of areas of isoceles triangles.... I'll explain on the board (or doc camera). You can find many discussions of this subject online; see, for example, https://en.wikipedia.org/wiki/Area_of_a_circle.

On Thursday, I hope that James will explain in more detail how

$$\int_a^b f(x)\,dx$$

is defined as a limit!!

Fixing $f(x)$ and the number a, we now change perspective and think of b as varying, rather than fixed. We consider

$$F(t) := \int_a^t f(x)\,dx,$$

so that b has been replaced by a new variable, t. It would be appropriate to call this integral an "indefinite integral" because the right endpoint t is not "definitive." It can vary!

For example, let $f(x) = \dfrac{1}{x}$ and take $a = 1$. Then

$$F(t) = \int_1^t \frac{1}{x}\,dx.$$

What function is this?

The answer to this particular question comes out of two points:

1. $F(1) = 0$; that's because the area from 1 to 1 under $1/x$ is 0.
2. The derivative of $F(t)$ (with respect to t) is $\dfrac{1}{t}$.

The second point is crazily non-obvious. It's the Fundamental Theorem of Calculus (in case $f(x) = \dfrac{1}{x}$).

Grant these two points. Then $F(t)$ has two properties that are shared by the function $\ln t$, namely vanishing at $t = 1$ and having the derivative $\dfrac{1}{t}$.

We then claim: $F(t) = \ln t$. The reason is that the difference $F(t) - \ln t$ has these two properties:

1. It's 0 at 1;
2. its derivative is identically 0.

I will try to convince you that a function whose derivative is 0 at all points is a constant. (It has zero slope everywhere.) If you grant this, then the value of the constant is 0 because it's 0 at 1. Because $F(t) - \ln t$ is then 0, we'll know $F(t) = \ln t$ as claimed.

Fundamental theorem of calculus

This is the crazily non-obvious point. In general it states that

$$\frac{d}{dt}\left(\int_a^t f(x)\,dx\right) = f(t)$$

when $f(x)$ is a continuous function on an interval including a and t.

In words: if you integrate $f(x)$ and differentiate the result, you get back to where you started.

Because we integrated from a and t and differentiated with respect to t, we ended up with $f(t)$ instead of $f(x)$.

However, lots of people write

$$\frac{d}{dx}\left(\int_a^x f(x)\,dx\right) = f(x)$$

and don't worry that x is used in two senses in the same equation.

How to prove that

$$\frac{d}{dt}\left(\int_a^t f(x)\,dx\right) = f(t)?$$

The proof is supplied by a picture that I will draw.

Summary so far

Suppose we want to calculate a definite integral $\int_a^b f(x)\,dx$.

We introduce $F(t) := \int_a^t f(x)\,dx$ so that

$$\int_a^b f(x)\,dx = F(b).$$

Next, imagine that we can find a function $G(t)$ whose derivative is $f(t)$. Then $G(t) - G(a)$ and $F(t)$ have the same derivatives; also they have the same value (namely, 0) at $t = a$. Hence they are equal. Thus

$$\int_a^b f(x)\,dx = F(b) = G(b) - G(a).$$

We write the expression $G(b) - G(a)$ as:

$$G(t)\Big]_a^b .$$

One Example

Let's calculate the area under $y = x^3$ between $x = 0$ and $x = 2$:

$$\int_0^2 x^3\,dx.$$

We know a function $G(t)$ whose derivative is t^3, namely $\dfrac{t^4}{4}$.
Hence

$$\int_0^2 x^3\,dx = \frac{t^4}{4}\Big]_0^2 = \frac{2^4}{4} = 4.$$

Another example

Calculate the area under $y = \sin x$ *from* $x = 0$ *to* $x = \pi$*.*

For this we need to find a function whose derivative is sin. We know that $\cos' = -\sin$, so $-\cos t$ has derivative $\sin t$. Hence

$$\int_0^\pi \sin x \, dx = (-\sin t)\Big]_0^\pi = -\cos(-\pi) + \cos(0) = 2.$$

To do calculations like this, we need to find functions of t whose derivatives are given functions $f(t)$. These functions are called *primitives* or *indefinite integrals*.

Usually people use x instead of t. They want to find a function $G(x)$ whose derivative is some given $f(x)$. The unknown function is written

$$G(x) = \int f(x)\,dx$$

and is called an indefinite integral. (We used the phrase "indefinite integral" before.)

When you have one, you include "$+C$" as part of the answer to indicate that you know that you can add an arbitrary constant to the primtive and still get something with the right derivative. For example, people write

$$\int \sin x\,dx = -\cos x + C.$$

We can calculate a lot of indefinite integrals because we know how to calculate many derivatives. For example:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

for $n \neq -1$, and

$$\int \sec x \, dx = \ln(\sec x + \tan x) + C,$$

the second formula coming from the first midterm.

On Thursday, James will begin to discuss the techniques that can be used to find primtives of more complicated functions $f(x)$.