This and that

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Math 10A
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First some announcements

The next midterm will be on Thursday, here, in 48 hours.

One two-sided sheet of notes, just like last time.

Comments and questions: see `piazza`.

James: “I’ve announced a review session on Piazza: tomorrow night 6–7:30 in 2 Leconte. The capacity is only 100 (that’s the best I could get!) so get there early. If you can’t get in, have a friend take notes for you. Possibly someone could post detailed (and legible) notes on Piazza afterward."

The next breakfast will be on Thursday, November 3 at 8:30AM. There are about a dozen people signed up, so there are roughly eight more slots.

To sign up, send me email.

Also, let's not forget the pop-in lunch on Friday at 12:30PM. The Faculty Club has chicken vindaloo as its curry this week.

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Nah.

Instead, I’ll talk about integration today. I’ll also call for questions.

Next week we’ll start our discussion of probability.

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I still think that it's worth discussing the fact that the area under the curve $y = e^{-x^2}$ is $\dfrac{\pi}{2}$. Let's try again. . . .

Set $A(t) = \left(\int_0^t e^{-x^2}\,dx\right)^2$. The assertion to be established is

$$A(\infty) \stackrel{?}{=} \frac{\pi}{4}, \text{ i.e., } \lim_{t\to\infty} A(t) = \frac{\pi}{4}.$$

Let

$$B(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx.$$

The method of proof is to show that $A(t) + B(t)$ has derivative equal to 0 (identically).

If you believe this, then you'll agree that $A(t) + B(t)$ is a constant. Taking $t = 0$, you see that the constant is $\dfrac{\pi}{4}$ because $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$.

Since $\lim\limits_{t\to\infty} B(t)$ is pretty obviously 0, we must have

$$\lim_{t\to\infty} A(t) = \frac{\pi}{4},$$

as desired.

In other words, we have to show $A'(t) = -B'(t)$. We do that by computing separately the derivatives of A and B.

First, A: The fundamental theorem of calculus gives

$$A'(t) = 2\left(\int_0^t e^{-x^2}\,dx\right)\cdot\frac{d}{dt}\left(\int_0^t e^{-x^2}\,dx\right) = 2e^{-t^2}\int_0^t e^{-x^2}\,dx.$$

In the integral on the right, put $u = \dfrac{x}{t}$, $x = tu$, $dx = t\,du$. The integral with respect to u runs from 0 to 1, and a short computation shows

$$A'(t) = \int_0^1 2te^{-t^2(1+u^2)}\,du.$$

Equivalently,

$$A'(t) = \int_0^1 2te^{-t^2(1+x^2)}\,dx;$$

we just changed the letter of the alphabet in the integration.

Recall that

$$B(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx.$$

To differentiate B with respect to t, all we have to do is differentate the integrand $\dfrac{e^{-t^2(1+x^2)}}{1+x^2}$ with respect to t and then integrate the result with respect to x.

However, the derivative of $\dfrac{e^{-t^2(1+x^2)}}{1+x^2}$ with respect to t is clearly $-2te^{-t^2(1+x^2)}$. This is exactly the negative of the function that you integrate to get $A'(t)$. Hence $A'(t) = -B'(t)$, which is what we wanted.