Applications of integration

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Math 10A

October 18, 2016

The homework changed

For the assignment due on Friday, the problems from §5.6 were inadvertently listed at first as coming from §5.7.

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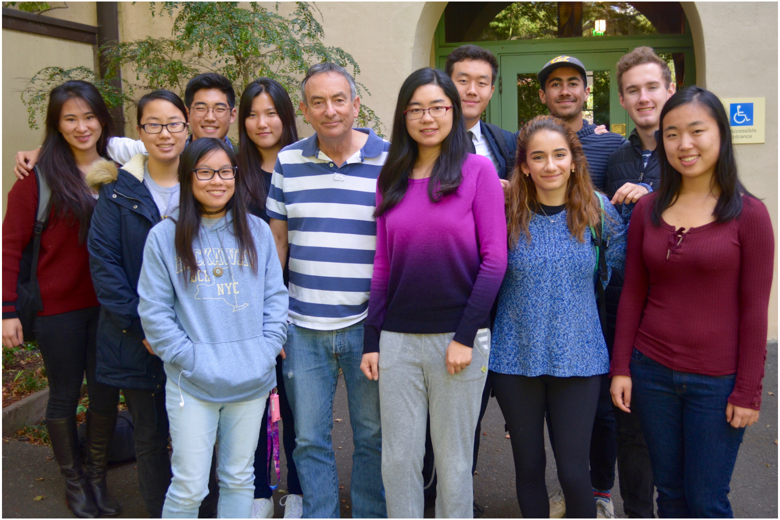
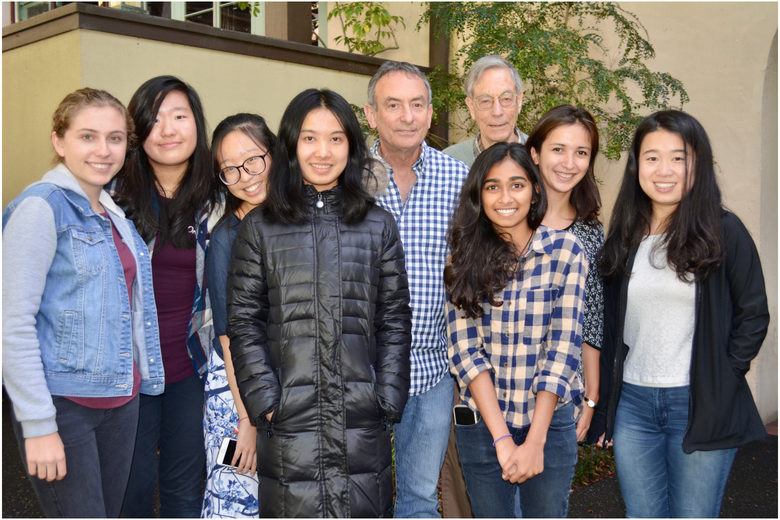
We had a fine breakfast on Thursday morning (before I left):



There are no further breakfasts planned right now.

We had a pop-in lunch yesterday:



The next pop-in lunch will be on Friday, October 28 at 12:30PM.

As you can all see, I am back on town. Unless there are further crises, I won't need to leave until November 17.

You all can suggest further breakfasts and lunches. I hope that you do! Extra points if you get up a group of co-conspirators ahead of time. If the proposed event is compatible with my schedule, I'll announce it to the class.

Today’s class period is about “applications of integration.” This phrase can embrace a broad range of topics. Please see especially §4 of the “Integral calculus” supplemental material.

A traditional calculus course includes applications like this:

- Calculating volumes of various solids (especially solids of revolution).
- Finding arc length.
- Figuring out surface area.
- Finding the center of mass of a 2- or 3-dimensional figure.
- Calculating areas of complicated regions in the plane.

What we'll do today (I think):

- Discuss the integral test and application to p-series (as implicitly requested on `piazza`;
- do an amazing calculation of the Gaussian integral;
- calculate some area or volume just to make contact with traditional courses.

Here is a fact that is super-important in probability:

$$\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}.$$

Equivalently, $\displaystyle\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi}$.

The most standard proof that Math 53 sophomores learn uses integrals over the plane and a transformation to polar coordinates. I am going to present what I hope is a relatively simple proof that everyone here can undertstand and retain. My reference is a handout that was written by my friend Keith Conrad. I will present Conrad's *third proof*.

Set $A(t) = \left(\int_0^t e^{-x^2}\,dx\right)^2$. The assertion to be established is

$$A(\infty) \stackrel{?}{=} \frac{\pi}{4}.$$

Using the fundamental theorem of calculus, we find

$$A'(t) = 2\left(\int_0^t e^{-x^2}\,dx\right) \cdot \frac{d}{dt}\left(\int_0^t e^{-x^2}\,dx\right) = 2e^{-t^2}\int_0^t e^{-x^2}\,dx.$$

In the integral on the right, put $u = \dfrac{x}{t}$, $x = tu$, $dx = t\,du$. The integral with respect to u runs from 0 to 1, and a short computation (chalkboard) shows

$$A'(t) = \int_0^1 2te^{-t^2(1+u^2)}\,du.$$

Let

$$B(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,dx.$$

Then $B'(t)$ is the integral from 0 to 1 of the derivative with respect to t of the integrand $\dfrac{e^{-t^2(1+x^2)}}{1+x^2}$. Computing again (chalkboard), we get $B'(t) = -A'(t)$.

As a result, there is a constant C such that $A(t) = -B(t) + C$. What is this constant?

Put $t = 0$: clearly $A(0) = 0$, while

$$B(0) = \int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1}(1) = \frac{\pi}{4}$$

(as we saw last Tuesday). Hence $C = \dfrac{\pi}{4}$. When $t \to \infty$, $B(t) \to 0$, so we end up with $A(\infty) = \dfrac{\pi}{4}$ as desired.

A confession: the function $B(t)$ is an integral with respect to x of the expression $\dfrac{e^{-t^2(1+x^2)}}{1+x^2}$, which depends on both x and t. We have used the fact that its derivative (with respect to t) is the integral with respect to x of the derivative of $\dfrac{e^{-t^2(1+x^2)}}{1+x^2}$ with respect to t. "The derivative of the integral is the integral of the derivative."

Scientists (i.e., end users) differentiate under the integral sign without worrying too much whether or not this is legal. Mathematicians are happy to justify the reasoning.