Math 10A midterm exam, October 27, 2016

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly, *USING WORDS* (not just symbols). Remember that the paper you hand in will be your only representative when your work is graded. Please write your name clearly on each page of your exam. Your paper will be scanned and will be processed using `Gradescope`. It is essential that you hand in all pages that you have received (including this cover sheet) and that the order of the pages be preserved.

Point counts:

Problem	1	2	3	4	5	Total
Points	9	8	8	7	8	40

These quick and dirty "solutions" were written by me (Ribet) just before the exam. These are not necessarily perfect, model solutions, but rather my attempt to explain what's going on.

You probably picked up on the fact that a number of these problems come from old exams or from the textbook's exercises. That's laziness on my part. To put a positive spin on my choice, we can say that I selected problems that you were likely to have seen before, thereby increasing the chances that you could do them without much strain.

1. Discuss the convergence of each of the following infinite series:

a. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{2^n}$,

If $a_n = \dfrac{\ln n}{2^n}$, then $\dfrac{a_{n+1}}{a_n} = \dfrac{\ln(n+1)}{\ln n} \cdot \dfrac{1}{2}$. By l'Hôpital's rule (or whatever), you can see that $\dfrac{\ln(n+1)}{\ln n} \longrightarrow 1$ as $n \to \infty$, so $\dfrac{\ln(n+1)}{\ln n} \longrightarrow \dfrac{1}{2} < 1$ and thus the series converges by the ratio test.

b. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$,

You acted with honesty, integrity, and respect for others.

The sum of the first three terms is

$$(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})=1-\frac{1}{4}.$$

In the same vein, the sum of the first N terms is $1-\dfrac{1}{N+1}$. As $N\to\infty$, this quantity approaches 1. Thus the series converges; its sum is 1.

c. $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{n!}$.

For each n, the fraction $\dfrac{n^n}{n!}$ is bigger than 1. Hence the nth term of this series does not approach 0. It follows that the series diverges. (The sum of the first N terms is bigger than N.)

2a. Express $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\left(1-\frac{2i}{n}\right)\left(\frac{2}{n}\right)$ as an integral of the form $\displaystyle\int_0^1 f(x)\,dx$.

This problem is a lot like problem 6 of §5.3, which was part of HW #7. In fact, it's problem 4 of §5.3. The idea is this: if you use right endpoints, then you see that

$$\int_0^1 f(x)\,dx=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}f]\left(\frac{i}{n}\right).$$

Matching things up, we see that we want

$$f\left(\frac{i}{n}\right)\stackrel{?}{=}2\cdot\left(1-\frac{2i}{n}\right).$$

To achieve this, we take

$$fx)=2(1-2x)=2-4x.$$

Then the limit is $\displaystyle\int_0^1(2-4x)\,dx$.

b. Express $\displaystyle\int_{-1}^{1}\cos x\,dx$ as a limit of Riemann sums.

There's no single correct answer to this problem. The problem, by the way, is #12 of §5.3 with the absolute value signs removed; James told me that the absolute value signs would be distracting. We can, for example, divide the interval $[-1, 1]$ into n equal segments and use left endpoints this time around. The intervals have length $\frac{2}{n}$, so the Riemann sum becomes

$$\frac{2}{n}\sum_{i=0}^{n-1}\cos\left(-1+\frac{2i}{n}\right).$$

The integral is the limit of this sum as n approaches ∞.

3a. Find the indefinite integral $\int \sin^3 x\,dx$. (Use that $\sin^2 + \cos^2 = 1$.)

The hint nudges us to write the integrand as $(\sin x)(1-\cos^2 x)$. Set $u = \cos x$, $du = -\sin x\,dx$. The integral is then

$$-\int (1-u^2)\,du = -u + \frac{u^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C.$$

b. Evaluate $\displaystyle\int_0^\infty \frac{e^x}{(1+e^x)^2}\,dx$.

Let $u = e^x$, $du = e^x\,dx$. In u-land, the integral becomes

$$\int_1^\infty \frac{1}{(1+u)^2}\,du = \left.\frac{-1}{1+u}\right]_1^\infty = -\left(0-\frac{1}{2}\right) = \frac{1}{2}.$$

4. Calculate the volume of the football-shaped solid obtained by rotating the interior of the ellipse $\frac{y^2}{4}+\frac{x^2}{9}=1$ about the x-axis.

We use the formula $\displaystyle\int_a^b \pi y^2\,dx$ for the volume. In this case $y^2 = 4\left(1-\frac{x^2}{9}\right)$. The possible values of x range from -3 to 3 because the quantity $1-\frac{x^2}{9}$ needs to be non-negative. The volume is

$$4\pi\int_{-3}^{3}\left(1-\frac{x^2}{9}\right)dx = 4\pi\left.\left(x-\frac{x^3}{27}\right)\right]_{-3}^{3} = 8\pi(3-1) = 16\pi.$$

5a. Evaluate this function of t: $\displaystyle\int_0^t se^{-s}\,ds$.

The integral is ripe for integration by parts. In the indefinite integral $\displaystyle\int se^{-s}\,ds$, we set $u = s$, $dv = e^{-s}\,ds$, $v = -e^{-s}$. Then the formula

$$\int u\,dv = uv - \int v\,du$$

becomes

$$\int se^{-s}\,ds = -se^{-s} + \int e^{-s}\,ds = -se^{-s} - e^{-s}.$$

The answer is then

$$(-se^{-s} - e^{-s})\Big]_0^t = 1 - te^{-t} - e^{-t}.$$

Alternatively, we can carry along the limits of integration as we work:

$$\int_0^t se^{-s}\,ds = -se^{-s}\Big]_0^t + \int_0^t e^{-s}\,ds = -te^{-t} + \int_0^t e^{-s}\,ds, \text{ etc.}$$

b. Find $\displaystyle\frac{d}{dt}\int_0^{t^2} e^{-x^2}\,dx$. (Hint: write the function to be differentiated in the form $G(t^2)$, where $G(u) = \displaystyle\int_0^u e^{-x^2}\,dx$.)

Following the hint, we write the function to be differentiated as $G(t^2)$. Use the chain rule: if $u = t^2$, the derivative is $G'(u)\dfrac{du}{dx}$. By the fundamental theorem of calculus, $G'(u) = e^{-u^2} = e^{-t^4}$; also $\dfrac{du}{dx} = 2t$. Thus the derivative appears to be $2te^{-t^4}$.