Math 10A final exam, December 16, 2016

Please put away all books, calculators, cell phones and other devices. You may consult a single two-sided sheet of notes. Please write carefully and clearly, *USING WORDS* (not just symbols). Remember that the paper you hand in will be your only representative when your work is graded. Please write your name clearly on each page of your exam. Your paper will be scanned and will be processed using `Gradescope`. It is essential that you hand in all pages that you have received (including this cover sheet) and that the order of the pages be preserved.

Point counts:

Problem	1	2	3	4	5	6	7	8	9	10	Total
Points	7	8	7	6	7	8	7	7	6	7	70

These quick and dirty "solutions" were written by me (Ribet) just before the exam. These are not necessarily perfect, model solutions, but rather my attempt to explain what's going on.

1a. Find all points on the interval $[0, 1]$ where the instantaneous rate of change of $f(x) = x^3 + x$ is equal to the average rate of change of $f(x)$ on the interval.

The "average rate of change" is defined on page 108 of Schreiber in a highlighted green box: it's $\dfrac{f(b) - f(a)}{b - a}$ if we're dealing with the interval $[a, b]$. Here the interval is $[0, 1]$, so it's $f(1)-f(0) = 2$. The instantaneous rate of change at c is $f'(c) = 3c^2 + 1$. We are asked to find all c in the interval such that $3c^2 + 1 = 2$, i.e., such that $c^2 = \dfrac{1}{3}$. There's only one such c, namely $\sqrt{\dfrac{1}{3}} = \dfrac{1}{\sqrt{3}}$.

b. If the derivative of $f(x)$ is $\dfrac{1}{x^2 + 1}$, what is the derivative of $f(x^{-1})$?

By the Chain Rule, $\dfrac{d}{dx} f(x^{-1}) = f'(x^{-1})\dfrac{d}{dx}(x^{-1}) = \dfrac{1}{x^{-2} + 1} \cdot \dfrac{-1}{x^2} = \dfrac{-1}{x^2 + 1}$. Thus the derivatives of $f(x)$ and $f(x^{-1})$ are negatives of each other.

This may seem strange at first. The explanation is that $f(x)$ is $\arctan x$ (plus a constant), so that $f(x^{-1})$ is the angle whose tangent is $\dfrac{1}{x}$. That angle is $\dfrac{\pi}{2} - f(x)$, so $f(x^{-1})$ is really a constant plus $-f(x)$.

You acted with honesty, integrity, and respect for others.

2a. Suppose that n is a positive integer. Calculate the integral

$$\int_1^n \ln x\, dx.$$

The fact that the upper limit of integration is an integer is a red herring—it could be any real number ≥ 1. To do this problem, you need to find an antiderivative of $\ln x$. This is a standard integration by parts situation, and the antiderivative will turn out to be $x \ln x - x$; see Example 2 on page 395 of Schreiber. (For the test, my intention was for you to do the integration by parts on your paper.) The definite integral is then

$$(x \ln x - x)\Big]_1^n = (n \ln n - n) - (1 \ln 1 - 1) = n \ln n - n + 1.$$

b. For what values of x is the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2(x+7)^n}{10^n(n+1)^2}$ convergent?

Let $a_n = \dfrac{n^2(x+7)^n}{10^n(n+1)^2}$. If $c = \dfrac{x+7}{10}$, then $a_n = \dfrac{n^2}{(n+1)^2}c^n$. Since $\dfrac{n^2}{(n+1)^2} \to 1$ (as $n \to \infty$), $a_n \to 0$ if and only if $c^n \to 0$. We know that $c^n \to 0$ if and only if $|c| < 1$. Hence the series *diverges* whenever $|c| \geq 1$. This condition translates to $|x + 7| \geq 10$; it holds precisely when $x \geq 3$ or $x \leq -17$.

It remains to see what happens when $-17 < x < 3$, or equivalently when $|c| < 1$. By the ratio test, the series $\sum a_n$ is convergent if $\displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ is less than 1. This limit is $|c|$, so we're in good shape: we learn that the series *converges* exactly in those situations where we didn't know that it diverges. To summarize: for all values of x, either the series diverges because its nth term doesn't approach 0 or it converges by the ratio test. A priori there might have been cases where the ratio $\left|\dfrac{a_{n+1}}{a_n}\right|$ approached 1 and the nth term approached 0. Then we would have to do further analysis to see whether or not the series converged in those cases. That would have been an added difficulty ("tricky problem"); maybe next year.

3. What approximation to $(1.02)^{1/2}$ is provided by the quadratic Taylor polynomial for $f(x) = x^{1/2}$ at the point $a = 1$? (Leave your answer as an unsimplified numerical expression.)

Basically this problem was an exercise in locating the relevant formula on your cheat sheet: the quadratic Taylor polynomial associated with $f(x)$ at $x = a$ is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2.$$

We have $a = 1$, $x = 1.02$, $x - a = 0.02$. Now $f(x) = \sqrt{x}$, so $f(1) = 1$; also $f'(x) = \frac{1}{2}x^{-1/2}$, so $f'(1) = \frac{1}{2}$. Similarly, $f''(1) = \frac{1}{2} \cdot \frac{-1}{2} = -\frac{1}{4}$. Hence the approximation in question is

$$1 + \frac{1}{2}(0.02) - \frac{1}{8}(0.0004) = 1.00995.$$

This approximation is correct to four decimal places.

4. Determine the volume of the solid obtained by revolving the area under the curve $y = x^2 + 1$ from $x = 0$ to $x = 2$ about the x-axis.

This is another problem where the main exercise is to consult your cheat sheet and find the right formula. The right formula here is $V = \int \pi y^2 \, dx$. In this specific case, we have

$$\pi \int_0^2 (x^2+1)^2 \, dx = \pi \int_0^2 (x^4 + 2x^2 + 1) \, dx.$$

An antiderivative of the integrand is $\frac{x^5}{5} + \frac{2}{3}x^3 + x$, so the value of the integral is $\frac{2^5}{5} + \frac{2}{3}2^3 + 2 = \frac{32}{5} + \frac{16}{3} + 2 = \frac{96+80+30}{15} = \frac{206}{15}$. Accordingly, the volume is $\frac{206\pi}{15}$.

5a. If a is a real number, calculate $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n$. (As for all problems on this exam, be sure to explain your reasoning with care.)

When I set up this problem at first, the real number was called A and I wrote $1+\dfrac{10A}{n}$ in place of $1+\dfrac{a}{n}$. I decided that the joke of having "$10A$" be part of the problem was not worth the extra burden on you.

We can write $\left(1+\dfrac{a}{n}\right)^n$ as $e^{\ln\left(\left(1+\frac{a}{n}\right)^n\right)}$. Hence if $L = \lim_{n\to\infty} \ln\left(1+\dfrac{a}{n}\right)^n$, then the limit to be calculated will be e^L. Now $\ln\left(1+\dfrac{a}{n}\right)^n = n\ln\left(1+\dfrac{a}{n}\right)$. Set $h=\dfrac{a}{n}$, so $h\to 0$ as $n\to\infty$. We have

$$L = \lim_{n\to\infty} n\ln\left(1+\frac{a}{n}\right) = \lim_{h\to 0} a\frac{\ln(1+h)}{h}$$

since $n=\dfrac{a}{h}$. The limit $\lim_{h\to 0}\dfrac{\ln(1+h)}{h}$ is 1 because it's (by definition) the derivative of the function $\ln(x)$ at $x=1$. (You can see this also by l'Hôpital's rule.) Hence $L=a$, and the answer to the question is e^a.

Why was this problem on the exam? Look for the sentence "This is a good exercise (or exam problem)" on the December 1 slides.

b. Let $f(x)=\dfrac{\ln x}{x}$ for $x>0$. What happens to $f(x)$ as the positive number x approaches 0?

The function $\ln x$ approaches $-\infty$ as $x\to 0+$. You're dividing a large negative number by x, which is tiny and positive. The quotient is, so to speak, even more large and negative. So $f(x)$ approaches $-\infty$ as the positive number x approaches 0. (Although the answer was more or less apparent, I hope that your answer included an explanation and was not simply the unsubstantiated statement that the function approaches $-\infty$.)

6a. Let $f(x)$ be the function $\dfrac{\ln x}{x}$, defined for x positive. Find $\lim_{x\to\infty} f(x)$.

Both x and $\ln x$ approach infinity as $x\to\infty$, so we can use l'Hôpital's rule to calculate the limit: the ratio of the derivatives is $\dfrac{1}{x}/1=\dfrac{1}{x}$, which approaches 0 as $x\to\infty$. Hence the limit is 0. Takeaway: $f(x)\to 0$ at the right end of its domain of definition, and $f(x)\to-\infty$ at the left end of its domain of definition (by problem **5b**). Hence the maximum value of $f(x)$ occurs "somewhere in the middle."

b. Does $f(x)$ have a global maximum value? If so, what is this value?

Yes, it has a maximum, which you find by setting the derivative equal to 0. The derivative is a fraction, whose denominator is x^2. The derivative vanishes when the numerator is 0; the numerator of the derivative is

$$x\frac{d}{dx}(\ln x) - (\ln x)\frac{d}{dx}(x) = 1 - \ln x.$$

The derivative is 0 exactly when $1 - \ln x = 0$, i.e., when $x = e$. The maximum value is then $f(e) = \frac{1}{e}$.

7. Which is more likely: getting 60 or more heads in 100 tosses of a fair coin or getting 225 or more heads in 400 tosses of a fair coin?

This is very similar to problems you've encountered before. We have to compute the z-statistic for the two situations. The statistic that's the furthest from the center is the less likely.

The general formula is $Z = \frac{(\overline{X} - \mu)\sqrt{N}}{\sigma}$. Here, $\mu = \frac{1}{2}$ and $\sigma = \frac{1}{2}$ as well. Of course, σ will be the same in both scenarios, so we can ignore it and just see which of the two expressions $(\overline{X} - \frac{1}{2})\sqrt{N}$ is bigger in absolute value. In the first scenario, $\overline{X}$ is replaced by 0.6 and $\sqrt{N} = 10$, so $(\overline{X} - \mu)\sqrt{N} = 0.1 \cdot 10 = 1$. In the second scenario, we have $\frac{25}{400} \times 20 = \frac{5}{4}$. Since $\frac{5}{4} > 1$, the second scenario is the less likely one. Thus it's more likely to get 60 heads in 100 tosses than to get 225 heads in 400 tosses.

8. If the continuous random variable X has PDF equal to $f(x)$, then we have

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

for all reasonable functions g. Use this information to calculate the expected value of $|X|$ when X is a standard normal variable (with mean 0 and standard deviation equal to 1).

According to the HW13 solutions file, the answer is $\sqrt{\dfrac{2}{\pi}}$. Let's see why. In the general description of the problem, the function g is the absolute value function; moreover, $f(x) = \dfrac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Consulting the description at the beginning of the statement of the problem, we realize that the number to be computed is $\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty} |x|e^{-x^2/2}\,dx$. Because the function being integrated is an even function, the integral in question is twice the analogous integral from 0 to ∞. Since $2 \cdot \dfrac{1}{\sqrt{2\pi}} = \sqrt{\dfrac{2}{\pi}}$, we now recognize that the answer in the HW solutions is correct if and only if we have

$$\int_0^\infty xe^{-x^2/2}\,dx = 1.$$

In the integral, set $u = x^2/2$, $du = x\,dx$. The integral becomes

$$\int_0^\infty e^{-u}\,du = -e^{-u}\Big]_0^\infty = 1,$$

as required.

9. Find all values of a and b such that

$$p(t) = \frac{ae^{bt}}{1+ae^{bt}}$$

is a cumulative distribution function.

The function $p(t)$ is a CDF if it has the following three properties: (1) it's 0 at $-\infty$; (2) it's 1 at $+\infty$; (3) it's a non-decreasing function. The last property means that $p(t) \le p(t')$ if $t \le t'$. To see how $p(t)$ behaves, we might multiply numerator and denominator by e^{-bt}. Then we see that

$$p(t) = \frac{a}{a+e^{-bt}}.$$

If $a = 0$, then $p(t)$ is identically 0, so it can't be a CDF. Let's assume that a is non-zero and divide numerator and denominator by a. If $c = 1/a$, then

$$p(t) = \frac{1}{1+ce^{-bt}}.$$

In order to get property (1), we need the denominator to get large (in absolute value) for $t \to -\infty$; this requires that b be positive. If b is positive, then $e^{-bt} \to 0$ as $t \to +\infty$, so (2) will be satisfied as well. For (3), we want ce^{-bt} to decrease as t increases. For that, we'd better have c positive because ce^{-bt} is decreasing in view of our assumption that b is positive. Note that c is positive if and only if a is positive because $c = 1/a$.

Conclusion: $p(t)$ is a CDF if and only if a and b are both positive.

10. Explain how the approximation
$$\int_1^n \ln x\,dx \approx \ln(n!) - \frac{1}{2}\ln n$$
can be obtained by averaging together left- and right-endpoint approximations to the integral. (Recall that $n! = 1 \cdot 2 \cdot 3 \cdots n$.)

[Problems 2a and 10 lead to *Stirling's approximation* to $n!$.]

The integral in question is the area under $y = \ln x$ between $x = 1$ and $x = n$. The segment of the x-axis from 1 to n can be divided into $n - 1$ segments, the first from 1 to 2 and the last one from 2 to n. If you use left-endpoints to approximate the area, then $\displaystyle\int_1^n \ln x\,dx$ is approximated by the sum of the areas of $n-1$ rectangles, all of which have width 1; their heights are $\ln 1, \ln 2, \ldots, \ln(n-1)$. The left-hand approximation is thus
$$\int_1^n \ln x\,dx \approx \ln 1 + \ln 2 + \cdots + \ln(n-1).$$
Similarly, using right endpoints, we get
$$\int_1^n \ln x\,dx \approx \ln 2 + \ln 3 + \cdots + \ln n.$$
If we average these two, we get this approximation ("trapezoidal rule"):
$$\int_1^n \ln x\,dx \approx \frac{1}{2}(\ln 1 + \ln n) + \ln 2 + \ln 3 + \cdots + \ln(n-1).$$
We can write the right-hand side as
$$\ln 1 + \ln 2 + \ln 3 + \cdots + \ln n - \frac{1}{2}(\ln 1 + \ln n).$$
The positive terms sum to $\ln(n!)$, since $n! = 1 \cdot 2 \cdot 3 \cdots n$ and since the log of a product is the sum of the logs. Furthermore, the term $\ln 1$ can be ignored, since it's 0. Thus the right-hand side is $\ln(n!) - \dfrac{1}{2}\ln n$.