SPACE VECTORS FORMING RATIONAL ANGLES

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In memory of John H. Conway

Abstract. We classify all sets of nonzero vectors in $\mathbb{R}^3$ such that the angle formed by each pair is a rational multiple of π. The special case of four-element subsets lets us classify all tetrahedra whose dihedral angles are multiples of π, solving a 1976 problem of Conway and Jones: there are 2 one-parameter families and 59 sporadic tetrahedra, all but three of which are related to either the icosidodecahedron or the B_3 root lattice. The proof requires the solution in roots of unity of a $W(D_6)$-symmetric polynomial equation with 105 monomials (the previous record was 21 monomials).

1. Introduction

1.1. **Rational-angle line configurations.** Call an angle *rational* if its degree measure is rational, or equivalently if its radian measure is in $\mathbb{Q}\pi$. Our main theorem classifies all sets S of nonzero vectors in $\mathbb{R}^3$ such that the angle formed by each pair is rational.

Scaling a nonzero vector $\mathbf{v}$ does not affect whether the angles it forms with other vectors are rational, so it is natural to consider the lines $\mathbb{R}\mathbf{v}$. In this paper, *line* means line in $\mathbb{R}^3$ through $\mathbf{0}$, and *plane* is defined similarly. A *rational-angle line configuration* is a set of lines such that each pair forms a rational angle. Call two configurations *equivalent* if there exists an orthogonal transformation mapping one to the other.

Example 1.1. Let $L \subset P$ be a line and plane. The set of lines in P forming a rational angle with L together with the line perpendicular to P is a rational-angle line configuration. Call it a *perpendicular configuration*. See the first image in Figure 1.

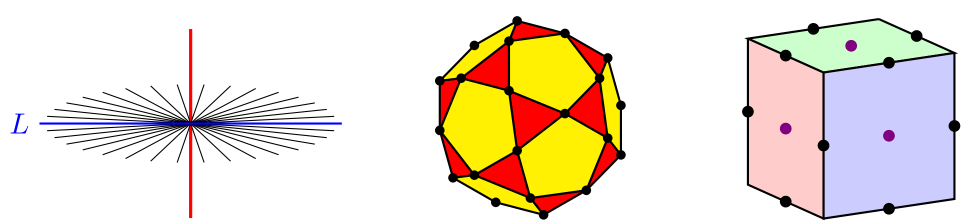

Figure 1. A perpendicular configuration, an icosidodecahedron, and the B_3 root system.

Any subset of a rational-angle line configuration is another, so it suffices to classify *maximal* rational-angle line configurations, those not contained in a strictly larger one. For $n < 4$, describing

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n	number of maximal rational-angle n-line configurations
$\aleph_0$	1
15	1
9	1
8	5
6	22, plus 5 one-parameter families
5	29, plus 2 one-parameter families
4	228, plus 10 one-parameter families and 2 two-parameter families
3	1 three-parameter family

Table 1. The number of maximal rational-angle line configurations with n lines, up to equivalence. For each n not shown, there are none. For the definition of family, see Definition 2.8. For a complete description of the families, see Section 11.

the rational-angle configurations of n lines is trivial since there are no *equations* that the angles between them must satisfy, only the obvious *inequalities*.

Theorem 1.2. *The maximal rational-angle line configurations, up to equivalence, fall into finitely many families and sporadic examples as enumerated in Table 1. In particular, each rational-angle line configuration not contained in a perpendicular configuration has at most* 15 *lines.*

Remark 1.3. Neither the finiteness in the first sentence nor the existence of *any* bound in the second sentence seems to follow from prior results in the literature. In fact, it seems that it was not even guessed that such bounds could exist. We ourselves will not know that such bounds exist, even in principle, until completing our entire argument.

Here are geometric descriptions of the three largest configurations:

Example 1.4. The $\aleph_0$-line configuration is the perpendicular configuration.

Example 1.5. The 15-line configuration consists of the lines connecting an icosidodecahedron's center to each of its 30 vertices. (The vertices of an icosidodecahedron are the midpoints of the edges of a regular icosahedron, or equivalently, the midpoints of the edges of a regular dodecahedron; see the second image in Figure 1.) The angles formed are all the multiples of $\pi/2$, $\pi/3$, $\pi/5$ in $(0, \pi)$.

Example 1.6. The 9-line configuration consists of the lines in the directions of the 18 roots of the B_3 root lattice (or equivalently, the C_3 root lattice, since the lengths are irrelevant). The angles formed are all the multiples of $\pi/3$ and $\pi/4$ in $(0, \pi)$. See the third image in Figure 1.

Some additional examples are described in Section 10.

Remark 1.7. The following problems are equivalent:

(a) classifying sets of nonzero vectors in $\mathbb{R}^3$ forming rational angles;
(b) classifying rational-angle line configurations;
(c) classifying *rational-angle plane configurations*, i.e., sets of planes such that each pair forms a rational angle (proof: take the perpendicular subspaces);
(d) classifying *spherical codes* with distances in $\mathbb{Q}\pi$, i.e., subsets of the unit sphere such that the spherical distance between any two points lies in $\mathbb{Q}\pi$ (proof: intersect each line with the sphere); and
(e) classifying convex polyhedra such that every two extended faces either form a rational angle or are parallel (for each rational-angle plane configuration $\mathscr{P}$ whose normal vectors span $\mathbb{R}^3$, choose closed half-spaces bounded by one or two planes parallel to each plane in $\mathscr{P}$, and consider their intersection, if bounded).

Therefore Theorem 1.2 solves all of them.

Remark 1.8. There exist polyhedra with rational dihedral angles having two extended faces meeting at an irrational angle outside the polyhedron. These we do not classify in general.

1.2. **Tetrahedra.** Call a tetrahedron *rational* if all six of its dihedral angles are rational. Rational tetrahedra have Dehn invariant 0, or equivalently are scissors-congruent to a cube [Deh01, Syd65], and as such are candidates for tetrahedra that can tile $\mathbb{R}^3$ [Deb80], the study of which dates back to Aristotle [Sen81]. Conway and Jones in 1976 called attention to the problem of classifying rational tetrahedra.[1] We solve the problem in Theorem 1.9 below.

A plane configuration is in *general position* if any three planes intersect in a point, or equivalently, if in the corresponding line configuration, no three lines are contained in any plane. Rational tetrahedra up to similarity are in bijection with rational-angle 4-plane configurations in general position up to equivalence: given a tetrahedron, take the plane through $\mathbf{0}$ parallel to each face. Because of this and Remark 1.7, Theorem 1.2 contains the classification of rational tetrahedra.

Given a tetrahedron with faces labeled $1, 2, 3, 4$, let α_{ij} be the dihedral angle formed by faces i and j, and list dihedral angles in the order $(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23})$ so as to pair each edge with the opposite edge.

Theorem 1.9. *The rational tetrahedra are those with dihedral angles*

$$(\pi/2,\ \pi/2,\ \pi-2x,\ \pi/3,\ x,\ x) \qquad \text{for } \pi/6 < x < \pi/2,$$
$$(5\pi/6-x,\ \pi/6+x,\ 2\pi/3-x,\ 2\pi/3-x,\ x,\ x) \qquad \text{for } \pi/6 < x \le \pi/3,$$

and the 59 *sporadic tetrahedra listed in Table 3. (Here, $x \in \mathbb{Q}\pi$ is assumed.)*

Remark 1.10. The first family in Theorem 1.9 was discovered in 1895 [Hil95, Art. 4]; see also [Had51] for a generalization to higher dimension and [MM18, §2] for an elegant calculation of its angles. The second family in Theorem 1.9 appears to be new.

Of the 59 sporadic rational tetrahedra, 15 (the tetrahedra $H_2(\pi/4)$, T_0–T_7, T_{13}, T_{16}–T_{18}, T_{21}, T_{23} in [Bol78, pp. 170–173]) were discovered between 1895 and 1974 [Hil95; Cox48, p. 192; Syd56; Gol58; Len62; Gol74], and the other 44 appear to be new.

Remark 1.11. We can "explain" almost all of the sporadic rational tetrahedra: Under the action of the Regge group $\mathfrak{R}$ (see Section 4), 56 of the 59 are equivalent to a tetrahedron coming from a 4-line subconfiguration of the 15- or 9-line configuration. The remaining three are in the $\mathfrak{R}$-orbit of the tetrahedron with dihedral angles $(\pi/7,\ 3\pi/7,\ \pi/3,\ \pi/3,\ 4\pi/7,\ 4\pi/7)$.

1.3. **Strategy of proof.** Geometry reduces the problem of determining rational-angle 4-line configurations to solving a polynomial equation whose variables are constrained to lie in the set μ of all roots of unity. There are two known methods for solving equations in roots of unity; one is practical for equations in up to 21 monomials, and the other is practical for equations in up to 3 variables, roughly. The complexity of each algorithm grows faster than exponentially.

What distinguishes our equation is that it has 105 monomials in 6 variables! To solve it, we need the key idea, never before used to solve equations in roots of unity in characteristic 0, of building upon work of Dvornicich and Zannier [DZ02] by working first in the quotient $\mathbb{Z}[\mu]/(2)$ of the subring $\mathbb{Z}[\mu] \subset \mathbb{C}$; this makes the problem barely doable:

1. Reducing modulo 2 yields a polynomial equation in $\mathbb{Z}[\mu]/(2)$ with only 12 monomials!

[1] Conway and Jones wrote "It seems quite probable that the general tetrahedron all of whose dihedral angles are rational can be found by our techniques" [CJ76, p. 239]. But we consider it unlikely that this is true — as explained in Section 1.3, our argument requires a combination of several different techniques, and the Conway–Jones method is only a small part of our proof.

2. We adapt the first method above to parametrize all solutions in μ to such equations in $\mathbb{Z}[\mu]/(2)$. This restricts the possible 6-tuples to lie in finitely many families, each parametrized by at most 3 variables.
3. Substituting each parametrization back into the original equation yields a polynomial equation (no longer mod 2) in at most 3 variables.
4. We solve each of these equations using the second method above.

Actually, we do not fully solve the equations as above, but we do enough to constrain the roots of unity in sporadic solutions to be of certain orders up to 840; then a large numerical computation, followed by an algebraic certification of results, handles these "small" cases. This yields a description of all 4-line configurations, in terms of 84696 parametrized families and sporadic examples of angle matrices recording the pairwise angles between vectors along the lines. These include the configurations corresponding to the tetrahedra in Theorem 1.9 but also many others in which at least three of the lines lie in a plane. Finally, the n-line configurations for $n = 5, 6, \ldots, 16$ in turn are determined by finding all $n \times n$ matrices for which each 4×4 principal submatrix belongs to one of the 84696 families; we employ an "early abort" strategy to avoid having to analyze $84696^{\binom{16}{4}}$ cases. The code for the various computations, written in `C++`, Magma [Magma], SageMath [Sage], and Singular [Sing], is available at `https://github.com/kedlaya/tetrahedra/`.

Remark 1.12. Dvornicich, Veneziano, and Zannier [DVZ20] study the rational angles formed by vectors in a lattice in $\mathbb{R}^2$. This leads to a problem of a different type, involving up to three variables constrained to be roots of unity, but also some variables constrained to be integers. Their analysis requires the determination of the rational points on some curves of genus ≥ 1.

Remark 1.13. Conway and Jones [CJ76, p. 239] claim that the "harder problem" of enumerating all tetrahedra scissors-congruent to a cube reduces via their method to an "ordinary diophantine equation". This does not seem correct, but nevertheless we expect that some of our new techniques might lead to progress on this problem, at least for tetrahedra satisfying additional constraints. As of now, it is not known that such tetrahedra form finitely many parametrized families, even in principle.

2. Realizability of angle matrices

Definition 2.1. If $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{R})$ and $I \subset \{1, \ldots, n\}$ with $|I| = m$, then $(a_{ij})_{i,j \in I} \in \mathrm{M}_m(\mathbb{R})$ is called an $m \times m$ *principal submatrix* of A. Its determinant is called a *principal minor* of A.

Given nonzero $\mathbf{v}, \mathbf{w} \in \mathbb{R}^d$, let $\angle \mathbf{vw} \in [0, \pi]$ be the radian measure of the angle they form. Let Σ^{d-1} be the unit sphere in $\mathbb{R}^d$; its elements are unit vectors. Let $\mathrm{M}_n(\mathbb{R})_0^{\mathrm{sym}}$ be the set of symmetric $n \times n$ matrices with diagonal entries equal to 0. Call $\Theta \in \mathrm{M}_n(\mathbb{R})$ *realizable in* $\mathbb{R}^d$ if it is in the image of

$$\begin{aligned} (\Sigma^{d-1})^n &\xrightarrow{\angle} \mathrm{M}_n(\mathbb{R})_0^{\mathrm{sym}} \\ (\mathbf{v}_1, \ldots, \mathbf{v}_n) &\longmapsto (\angle \mathbf{v}_i \mathbf{v}_j). \end{aligned}$$

Proposition 2.2. *Suppose that* $\Theta = (\theta_{ij}) \in \mathrm{M}_n(\mathbb{R})_0^{\mathrm{sym}}$ *has entries in* $[0, \pi]$*. Let* $C = (\cos \theta_{ij})$*. Then* Θ *is realizable in* $\mathbb{R}^d$ *if and only if*

1. *for every* $m \leq d$*, each* $m \times m$ *principal minor of* C *is nonnegative, and*
2. *each* $(d+1) \times (d+1)$ *principal minor of* C *equals* 0.

Proof. See the proof of Lemma 2.1 of [BG17]. □

Corollary 2.3. *Let* $\Theta \in \mathrm{M}_n(\mathbb{R})$ *for some* $n \geq d+1$*. Then* Θ *is realizable in* $\mathbb{R}^d$ *if and only if every* $(d+1) \times (d+1)$ *principal submatrix of* Θ *is realizable in* $\mathbb{R}^d$*.*

Remark 2.4. The 1×1 and 2×2 principal minors of C in Proposition 2.2 are 1 and $1 - \cos^2 \theta_{ij}$, which are automatically nonnegative.

Remark 2.5. The nonnegative real numbers α, β, γ are sides of a possibly degenerate spherical triangle if and only if $\alpha \le \beta + \gamma,\ \ \beta \le \gamma + \alpha,\ \ \gamma \le \alpha + \beta$, and $\alpha + \beta + \gamma \le 2\pi$. Therefore, such angle inequalities give the condition for a 3×3 principal minor of C as in Proposition 2.2 to be nonnegative.

Let $\mathcal{P}_n \subset \mathrm{M}_n(\mathbb{R})_0^{\mathrm{sym}}$ be the polytope defined by the $4\binom{n}{3}$ inequalities, four as in Remark 2.5 from each of the 3×3 principal submatrices of Θ. Let $\mathcal{H}_n \subset \mathrm{M}_n(\mathbb{R})_0^{\mathrm{sym}}$ be the analytic subvariety defined by the vanishing of the determinants of the 4×4 Gram matrices $(\cos \theta_{ij})_{i,j \in I}$, one for each 4-element subset $I \subset \{1, \ldots, n\}$.

Corollary 2.6. *The set of* $\Theta \in \mathrm{M}_n(\mathbb{R})$ *realizable in* $\mathbb{R}^3$ *is* $\mathcal{P}_n \cap \mathcal{H}_n$.

Proof. Combine the $d = 3$ case of Proposition 2.2 with Remarks 2.4 and 2.5. ☐

Definition 2.7.

(i) A *family of* $\mathbb{R}^3$*-realizable* $n \times n$ *rational-angle matrices* is a polytope Q contained in $\mathcal{P}_n \cap \mathcal{H}_n$ such that
  - the vertices of Q are matrices with entries in $\mathbb{Q}\pi$.
  - some element of Q has no off-diagonal angles equal to 0 or π; and
  - Q is not strictly contained in another polytope satisfying these conditions.

(ii) The *number of parameters* of the family is the dimension of Q.

(iii) Call Q *maximal* if there is no family Q' of $\mathbb{R}^3$-realizable $(n+1) \times (n+1)$ rational-angle matrices such that Q equals the set of upper left principal submatrices of the matrices in Q'.

Definition 2.8. An *r-parameter family of rational-angle line configurations* is the set of line configurations represented by all matrices with entries in $\mathbb{Q}\pi$ belonging to a particular polytope Q as described in Definition 2.7.

3. Subvarieties of algebraic tori

Identify $\mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}}$ with $\mathbb{R}^6$ via $\Theta \mapsto (\theta_{12}, \theta_{34}, \theta_{13}, \theta_{24}, \theta_{14}, \theta_{23})$. Let $\mathcal{P} = \mathcal{P}_4 \subset [0, \pi]^6$. Let $\mathcal{H} = \mathcal{H}_4 \subset \mathbb{R}^6$; it is the analytic hypersurface

$$\det\,(\cos \theta_{ij})_{1 \le i,j \le 4} = 0. \tag{1}$$

Expanding (1) and substituting $\cos \theta = (e^{\mathrm{i}\theta} + e^{-\mathrm{i}\theta})/2$ yields the six-variable equation

$$-20 + 4 \sum z_{12}^{\pm 1} z_{13}^{\pm 1} z_{23}^{\pm 1} - 2 \sum z_{12}^{\pm 2} - 2 \sum z_{12}^{\pm 1} z_{13}^{\pm 1} z_{24}^{\pm 1} z_{34}^{\pm 1} + \sum z_{12}^{\pm 2} z_{34}^{\pm 2} = 0 \tag{2}$$

in which each sum ranges over the S_4-orbit of each monomial and over all possible choices of signs. The number of monomials is $1 + 4 \cdot 2^3 + 6 \cdot 2^1 + 3 \cdot 2^4 + 3 \cdot 2^2 = 105$.

Let Z be the subvariety of the algebraic torus $\mathbb{G}_m^6$ over $\mathbb{Q}$ defined by (2). Let $\exp\colon \mathbb{R}^6 \to \mathbb{G}_m^6(\mathbb{C})$ be the map applying $\theta \mapsto e^{\mathrm{i}\theta}$ to each coordinate, so $\mathcal{H} = \exp^{-1}(Z(\mathbb{C}))$. The monomials appearing in (2) generate an index-8 subgroup Λ of the group of all Laurent monomials in the z_{ij}; let T be the torus whose coordinate ring is their span. Thus there is an isogeny $\tau\colon \mathbb{G}_m^6 \to T$ and a closed subvariety $Y \subset T$ such that $Z = \tau^{-1} Y$. The kernel of τ is the elementary abelian group of order 8 consisting of $(z_{ij}) \in \{\pm 1\}^6$ such that $z_{ij} z_{jk} z_{ik} = 1$ for all $i < j < k$. To summarize, we have a cartesian diagram of spaces

$$\begin{tikzcd}
 & \mathcal{H} \arrow[r] \arrow[d, hook] & Z(\mathbb{C}) \arrow[r, two heads] \arrow[d, hook] & Y(\mathbb{C}) \arrow[d, hook] \\
(\Sigma^2)^4 \arrow[r, "\angle"] & \mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}} \simeq \mathbb{R}^6 \arrow[r, "\exp"] & \mathbb{G}_m^6(\mathbb{C}) \arrow[r, two heads, "\tau"] & T(\mathbb{C})
\end{tikzcd} \tag{3}$$

For an abelian group G, let G_{tors} be its torsion subgroup. The following problems are equivalent:

1. Determine all rational-angle 4-line configurations.
2. Determine $\mathcal{P} \cap \mathcal{H} \cap (\mathbb{Q}\pi)^6$. (Here we use the $n = 4$ case of Corollary 2.6.)
3. Determine $Z(\mathbb{C}) \cap \mu^6$. (We have $\theta \in \mathbb{Q}\pi$ if and only if $e^{\mathrm{i}\theta} \in \mu$. We dropped the inequalities defining $\mathcal{P}$, but these are easy to impose at the end of the computation.)
4. Determine $Y(\mathbb{C}) \cap T(\mathbb{C})_{\mathrm{tors}}$.

To solve 1, we will solve 3, but we will also use that $Z = \tau^{-1}Y$ and that Y has additional symmetry described in the next section.

4. Regge symmetry

The signed permutation group $S_n^{\pm} := S_n \ltimes \{\pm 1\}^n$ acts on $(\Sigma^{d-1})^n$ by permuting and negating the n vectors. Similarly, S_n acts on $\mathrm{M}_n(\mathbb{R})_0^{\mathrm{sym}}$ by simultaneously permuting rows and columns, and the ith generator of $\{\pm 1\}^n$ acts affine-linearly by applying $x \mapsto \pi - x$ to each entry of the ith row and ith column except the (i,i) entry. The element $(-1, \ldots, -1)$ acts trivially on $\mathrm{M}_n(\mathbb{R})_0^{\mathrm{sym}}$.

Now let $n = 4$ and $d = 3$. The $S_4^{\pm}$-action on $\mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}}$ is compatible with algebraic actions of $S_4^{\pm}$ on $\mathbb{G}_m^6$ (not fixing 1) and T (fixing 1) such that the maps in the bottom row of (3) are $S_4^{\pm}$-equivariant.

The $S_4^{\pm}$-action on $\mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}}$ preserves $\mathcal{H}$ and $\mathcal{P}$. Surprisingly, there is a larger group that preserves $\mathcal{H}$ and $\mathcal{P}$, coming from exotic symmetries of the space of labeled tetrahedra, as we will explain.

Fix an unordered partition of $\{1,2,3,4\}$ into pairs, say $\{\{1,2\},\{3,4\}\}$, which we abbreviate as 12,34. Following [Reg59], let $r = r_{12,34}$ be the linear operator on $\mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}} \simeq \mathbb{R}^6$ sending (x_{ij}) to (x'_{ij}) where $x'_{12} := x_{12}$, $x'_{34} := x_{34}$, and $x'_{ij} := s - x'_{ij}$ for all other $i < j$, where $s := (x_{13}+x_{24}+x_{14}+x_{23})/2$.

Let $\Delta \subset \mathbb{R}^3$ be a *labeled* tetrahedron; labeled means that the faces are numbered $1,2,3,4$. For each $i \neq j$, let e_{ij} be the edge formed by intersecting faces i and j, let ℓ_{ij} be the length of e_{ij}, and let α_{ij} be the dihedral angle along e_{ij}. Define $L_\Delta = (\ell_{ij})$ and $A_\Delta = (\alpha_{ij})$; both are in $\mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}}$.

Theorem 4.1 (Ponzano and Regge)**.** *For each labeled tetrahedron Δ, there exists a labeled tetrahedron Δ', unique up to congruence, such that $L_{\Delta'} = rL_\Delta$ and $A_{\Delta'} = rA_\Delta$. Moreover, Δ and Δ' are scissors-congruent.*

Proof. The first statement was proved in [PR68, Appendices B and D] by a brute force calculation. Geometric proofs have recently been discovered [AI19, Rud19], but they are not simple. The scissors congruence was first observed in [Rob99, Corollary 10]. □

Remark 4.2. The same theorem holds in spherical and hyperbolic geometry: see [Moh03, TW05, AI19, Rud19].

Definition 4.3. Call the operator $r = r_{12,34}$ and its analogues $r_{13,24}$ and $r_{14,23}$ *Regge operators*. Together with S_4, they generate a subgroup $\mathfrak{R} \subset \mathrm{GL}(\mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}}) \simeq \mathrm{GL}_6(\mathbb{R})$; in fact, r and S_4 already generate $\mathfrak{R}$ since the other Regge operators are S_4-conjugates of r. The group $\mathfrak{R}$ is isomorphic to $S_4 \times S_3$ [Reg59], but the isomorphism sends the original S_4 to the graph of a surjection $S_4 \to S_3$, not a normal subgroup, let alone a direct factor. Let $\mathfrak{R}^{\pm}$ be the subgroup of the affine linear group of $\mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}}$ generated by the image of $S_4^{\pm}$ and the Regge operators. Then $|\mathfrak{R}| = 2^4 3^2$ and $|\mathfrak{R}^{\pm}| = 2^7 3^2$.

Identify the z_{ij} with the standard basis of $\mathbb{Z}^6$, but scale the Euclidean norm so that $\langle z_{ij}, z_{ij} \rangle = 1/2$. Then Λ is a lattice. For each $c \in \mathbb{Z}$, let $\Lambda_c \subset \Lambda$ be the set of monomials in (2) with coefficient c. Checking inner products shows that Λ_{-2} is a copy of the D_6 root system! Let $W(D_6)$ be the Weyl group, which we view as acting on the right on Λ, so that it acts on the left on T. For each $c \in \{-20, 4, -2, 1\}$, the set Λ_c is a $W(D_6)$-orbit, so $W(D_6)$ preserves Y. By the theory of representations of reductive groups (see [Mil17, Theorem 22.38]), it follows that the left side of (2) is the character of a virtual representation of the algebraic group Spin_{12}.

The $S_4^{\pm}$-action on T preserves the norm on Λ, so it factors through $W(D_6)$. A brief calculation shows that the action of r on $\mathrm{M}_4(\mathbb{C})_0^{\mathrm{sym}}$ corresponds to a linear action on $\mathbb{Q}^6 \simeq \Lambda \otimes \mathbb{Q}$ that preserves Λ_{-2} and hence is in $W(D_6)$, so the homomorphism $S_4^{\pm} \to W(D_6)$ extends to $\mathfrak{R}^{\pm} \to W(D_6)$. Since (3) is cartesian, $\mathfrak{R}^{\pm}$ preserves $\mathcal{H}$.

In summary, we have a two-row cartesian diagram of spaces and a sequence of homomorphisms of groups, each acting on the spaces above it, compatibly with respect to the homomorphisms:

$$\begin{tikzcd}
 & \mathcal{H} \arrow[r] \arrow[d, hook] & Z(\mathbb{C}) \arrow[r, two heads] \arrow[d, hook] & Y(\mathbb{C}) \arrow[d, hook] \\
(\Sigma^2)^4 \arrow[r, "\angle"] & \mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}} \simeq \mathbb{R}^6 \arrow[r, "\exp"] & \mathbb{G}_m^6(\mathbb{C}) \arrow[r, two heads, "\tau"] & T(\mathbb{C}) \\
S_4^{\pm} \arrow[r] & \mathfrak{R}^{\pm} \arrow[rr, hook] & & W(D_6).
\end{tikzcd} \tag{4}$$

Finally, $S_4^{\pm}$ preserves $\mathcal{P}$, and direct calculation shows that r does too, so $\mathfrak{R}^{\pm}$ preserves $\mathcal{P}$. This extra symmetry will simplify our calculations and the statements of our results.

5. Cyclotomic relations

Recall from the end of Section 3 that we need to find the torsion points on a hypersurface Z in a torus $\mathbb{G}_m^6$; this amounts to solving (2) in roots of unity. Prior to our work, there were two general approaches to solving such problems:

- Classify integer relations involving few roots of unity (this section).
- Use the Galois theory of cyclotomic fields and induction on the dimension (Section 7).

But, crucially, we develop also a new method in Section 6 involving cyclotomic relations modulo 2. We need all three methods to find the torsion points on our particular variety Z; see Section 9.

The classification of additive relations among roots of unity grows out of work of Gordan [Gor77], de Bruijn [dB53], Rédei [Réd59, Réd60], and Schoenberg [Sch64]. Relations among n roots of unity have been classified for $n \le 7$ by Mann [Man65], $n \le 8$ by Włodarski [Wło69], $n \le 9$ by Conway and Jones [CJ76, Theorem 6], $n \le 12$ by Poonen and Rubinstein [PR98, Theorem 3.1], $n \le 21$ by Christie, Dykema, and Klep [CDK20], and partially for $n \le 24$ by Fu [Fu19]. The case $n \le 12$ has the following consequence.

Theorem 5.1. *Let $x_1, \ldots, x_n \in \mathbb{Q}$ be a sequence with $n \le 6$ such that $\sum_{i=1}^n \cos(2\pi x_i) = 0$, but no nonempty proper subsequence has the same property. Then $x_1, \ldots, x_n$ can be obtained from one of the sequences in Table 2 by some combination of permutation of terms, individual negation, individual addition of integers, and simultaneous addition of* $1/2$.

Proof. Combine Theorem 3.1, Lemma 4.1, and Lemma 4.2 of [PR98]. □

Remark 5.2. Building on these ideas, algorithms for finding the solutions of a polynomial equation in roots of unity have been described by Sarnak and Adams [SA94]; Filaseta, Granville, and Schinzel [FGS08]; and Leroux [Ler12]. But these algorithms scale exponentially in the number of variables and the number of monomials, so executing them on a polynomial with 105 monomials, as in (2), is infeasible.

6. Mod 2 cyclotomic relations

Let $\overline{\mu}$ be the image of μ in $\mathbb{Z}[\mu]/(2)$, so $\overline{\mu} \simeq \mu/\{\pm 1\}$. (We would lose too much information if instead we chose a prime $\mathfrak{p}$ above 2 and worked in the residue field $\mathbb{Z}[\mu]/\mathfrak{p} \simeq \overline{\mathbb{F}}_2$.) For $n \ge 1$, let $\mu_n := \{z \in \mu : z^n = 1\}$, let $\overline{\mu}_n$ be the image of μ_n in $\mathbb{Z}[\mu]/(2)$, and let $\zeta_n := e^{2\pi \mathrm{i}/n} \in \mu_n$.

Length	Type	Values
1	n/a	$\frac{1}{4}$
2^*	$2R_2$	$x + (0, h)$
3	$(R_5 : R_3)$	$\frac{1}{3} + h, \frac{1}{5}, \frac{2}{5}$
3^*	$2R_3$	$x + (0, \frac{1}{3}, \frac{2}{3})$
4	$(R_5 : 3R_3)$	$\frac{1}{3}, \frac{1}{3} + \frac{1}{5}, \frac{2}{3} + \frac{1}{5}, \frac{2}{5} + h$
		$\frac{1}{3}, \frac{1}{3} + \frac{2}{5}, \frac{2}{3} + \frac{2}{5}, \frac{1}{5} + h$
	$(R_7 : R_3)$	$\frac{1}{3} + h, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}$
5	$(R_7 : 3R_3)$	$\frac{1}{3}, \frac{1}{3} + \frac{1}{7}, \frac{2}{3} + \frac{1}{7}, \frac{2}{7} + h, \frac{3}{7} + h$
		$\frac{1}{3}, \frac{1}{3} + \frac{2}{7}, \frac{2}{3} + \frac{2}{7}, \frac{1}{7} + h, \frac{3}{7} + h$
		$\frac{1}{3}, \frac{1}{3} + \frac{3}{7}, \frac{2}{3} + \frac{3}{7}, \frac{1}{7} + h, \frac{2}{7} + h$
	$(R_7 : R_5)$	$\frac{1}{5} + h, \frac{2}{5} + h, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}$
5^*	$2R_5$	$x + (0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$
6	$(R_7 : 5R_3)$	$\frac{1}{3}, \frac{1}{3} + \frac{1}{7}, \frac{1}{3} + \frac{2}{7}, \frac{2}{3} + \frac{1}{7}, \frac{2}{3} + \frac{2}{7}, \frac{3}{7} + h$
		$\frac{1}{3}, \frac{1}{3} + \frac{1}{7}, \frac{1}{3} + \frac{3}{7}, \frac{2}{3} + \frac{1}{7}, \frac{2}{3} + \frac{3}{7}, \frac{2}{7} + h$
		$\frac{1}{3}, \frac{1}{3} + \frac{2}{7}, \frac{1}{3} + \frac{3}{7}, \frac{2}{3} + \frac{2}{7}, \frac{2}{3} + \frac{3}{7}, \frac{1}{7} + h$
	$(R_7 : R_5, 2R_3)$	$\frac{1}{5}, \frac{2}{5}, \frac{1}{3} + \frac{1}{7}, \frac{2}{3} + \frac{1}{7}, \frac{2}{7} + h, \frac{3}{7} + h$
		$\frac{1}{5}, \frac{2}{5}, \frac{1}{3} + \frac{2}{7}, \frac{2}{3} + \frac{2}{7}, \frac{1}{7} + h, \frac{3}{7} + h$
		$\frac{1}{5}, \frac{2}{5}, \frac{1}{3} + \frac{3}{7}, \frac{2}{3} + \frac{3}{7}, \frac{1}{7} + h, \frac{2}{7} + h$
	$(R_7 : (R_5 : 2R_3))$	$\frac{2}{5} + h, \frac{1}{3} + \frac{1}{5}, \frac{2}{3} + \frac{1}{5}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}$
		$\frac{1}{5} + h, \frac{1}{3} + \frac{2}{5}, \frac{1}{3} + \frac{2}{5}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}$
	$(R_{11} : R_3)$	$\frac{1}{3} + h, \frac{1}{11}, \frac{2}{11}, \frac{3}{11}, \frac{4}{11}, \frac{5}{11}$
6^*	$2(R_5 : R_3)$	$x + (\frac{1}{3} + h, \frac{2}{3} + h, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})$

TABLE 2. Indecomposable additive relations among at most 6 cosines of rational multiples of 2π, up to transformations listed in Theorem 5.1. The symbol h stands for $\frac{1}{2}$. A length of n^* indicates a shift by an auxiliary parameter $x \in \mathbb{Q}$. The type is notated as per [PR98, Table 3.1 and Table 3.2]. By Theorem 6.10, this table (with one addition) also describes mod 2 cosine relations; in these, we may ignore shifts by h.

By a *mod* 2 *relation*, we mean a finite subset $S \subset \overline{\mu}$ summing to 0 in $\mathbb{Z}[\mu]/(2)$. Call S *indecomposable* if $S \neq \emptyset$ and S is not the disjoint union of two nonempty relations. The *weight* of S is $w(S) := |S|$. The *level* $\ell(S)$ is the smallest $n \geq 1$ such that $S \subset \overline{\mu}_n$. Call relations S and S' *equivalent* if $S' = \lambda S$ for some $\lambda \in \overline{\mu}$. Call S *minimal* if $\ell(S) \leq \ell(S')$ for all S' equivalent to S.

The goal of this section is Theorem 6.10, the mod 2 analogue of Theorem 5.1. We follow the proof of [PR98, Theorem 3.1]. First we establish an analogue of [CJ76, Theorem 1]:

Lemma 6.1. *The level of any minimal indecomposable mod* 2 *relation is odd and squarefree.*

Proof. Let S be the relation. Let $N = \ell(S)$. Suppose that p is a prime such that p^2 divides N. Since

$$\mathbb{Z}[\zeta_N] = \mathbb{Z}[\zeta_{N/p}][T]/(T^p - \zeta_{N/p})$$

and similarly after reduction mod 2, the intersection of S with each coset of the group $\overline{\mu}_{N/p}$ is another relation. Since S is indecomposable, it is contained in a single coset, so S is equivalent to a relation of level dividing N/p, a contradiction. Thus N is squarefree. If N is even, then $\overline{\mu}_N = \overline{\mu}_{N/2}$, so $\ell(S) \neq N$. □

Definition 6.2. For each odd prime p, let R_p denote the set $\overline{\mu}_p$ viewed as a weight p relation.

Let $\oplus$ denote symmetric difference of sets.

Definition 6.3. For relations $S, T_1, \ldots, T_j$, let $(S : T_1, \ldots, T_j)$ denote any relation of the form $S' \oplus T_1' \oplus \cdots \oplus T_j'$, where $S', T_1', \ldots, T_j'$ are equivalent to $S, T_1, \ldots, T_j$, respectively; $\#(S' \cap T_i') = 1$ for all i; and $T_i' \cap T_k' = \emptyset$ whenever $i \neq k$. (The equivalence class of such a relation need not be determined by the equivalence classes of $S, T_1, \ldots, T_j$.)

The following is an analogue of [PR98, Lemma 3.4], and, by extension, of [CJ76, Theorem 5]. A direct analogue of the latter result, working modulo any prime, can be found in [DZ02].

Lemma 6.4. *Let S be a minimal indecomposable mod 2 relation of level pM, where $p \nmid M$. If S intersects some coset of $\overline{\mu}_M$ in at most one element, then S is of the form $(R_p : T_1, \ldots, T_j)$, where $0 \leq j < p$, each T_i is nonempty with $\ell(T_i) \mid M$, and*

$$\sum_{i=1}^{j} (w(T_i) - 2) = w(S) - p. \tag{5}$$

Proof. Reducing

$$\mathbb{Z}[\zeta_{pM}] = \mathbb{Z}[\zeta_M][T]/(T^{p-1} + \cdots + T + 1),$$

modulo (2) shows that the intersections of S with the cosets of $\overline{\mu}_M$ must have sums which are rotations of each other by powers of ζ_p. No such intersection can be empty, or else each intersection would itself be a relation, equivalent to one of level dividing M, contradicting the hypotheses on S. Therefore some intersection has one element. Then each of the other intersections is either itself a singleton set or the complement of a single root of unity in some mod 2 relation. □

Corollary 6.5. *Each minimal indecomposable mod 2 relation S of weight at most 5 is R_3 or R_5.*

Proof. Let p be the largest prime dividing $\ell(S)$. In Lemma 6.4, $w(T_i) \geq 3$ for each i, so $w(S) \geq p$, with equality if and only if $S = R_p$. If $p = 3$, then $w(S) \leq \ell(S) = 3 = p$; if $p \geq 5$, then $w(S) \leq 5 \leq p$. Thus the equality holds, with $p = 3$ or $p = 5$. □

Lemma 6.6. *Let S be a minimal indecomposable mod 2 relation of level pM with $p \nmid M$. Then the intersections of S with the cosets of $\overline{\mu}_M$ cannot all have exactly two elements.*

Proof. As in the proof of Lemma 6.4, $S \cap \zeta_p^i \mu_m = \zeta_p^i U_i$ for some two-element sets $U_0, \ldots, U_{p-1}$ which all have equal sum. By Corollary 6.5, each $U_i \oplus U_j$ must be empty, so $U_0 = \cdots = U_{p-1}$. Then S is the union of two relations of type R_p, contradicting indecomposability. □

Theorem 6.7. *For each $w \in \{3, \ldots, 12\}$, the indecomposable mod 2 relations of weight w are precisely the mod 2 reductions of the indecomposable relations of weight w listed in* [PR98, Table 3.1].

(The statement of Theorem 6.7 must exclude $w = 2$, because R_2 reduces mod 2 to the empty relation.)

Proof of Theorem 6.7. Let S be a minimal indecomposable mod 2 relation of level N and weight w. By Lemma 6.1, we can write $N = p_1 \cdots p_s$ where $2 < p_1 < \cdots < p_s$. By Lemma 6.4, $p_s \leq w \leq 12$, so $p_s \leq 11$.

- Suppose that $p_s = 3$. Then Lemma 6.4 yields $S = R_3$.

- Suppose that $p_s = 5$. Each coset of $\overline{\mu}_3$ is itself a relation of weight 3, so the intersection of S with any such coset has at most two elements. By Lemma 6.6, the intersections cannot all have exactly two elements, so Lemma 6.4 yields $S = (R_5 : jR_3)$ for some $j \in \{0, \dots, 4\}$.
- Suppose that $p_s = 7$. Then Lemma 6.4 implies that $S = (R_7 : T_1, \dots, T_j)$ for some j with $\sum_{i=1}^{j} (w(T_i) - 2) \leq 5$. By the previous step, each T_i must have one of the forms R_3, R_5, $(R_5 : R_3)$, $(R_5 : 2R_3)$, which have $w(T_i) - 2$ being 1, 3, 4, 5, respectively. By considering the partitions of 5 into parts of these sizes, we obtain relations of the indicated forms.
- Suppose that $p_s = 11$. Then Lemma 6.4 yields $S = R_{11}$ or $S = (R_{11} : R_3)$. $\square$

Corollary 6.8. *Every mod 2 relation of weight at most 12 is the reduction of a genuine cyclotomic relation (i.e., a subset of μ summing to 0 in $\mathbb{Z}[\mu]$) of the same weight.*

Proof. Reduce to the indecomposable case and apply Theorem 6.7. $\square$

To pass from mod 2 cyclotomic relations to cosine relations, we argue as in [PR98, Lemma 4.1]. Keep in mind that the decomposition of a mod 2 relation into indecomposable relations is not *a priori* guaranteed to be unique.

Lemma 6.9. *Let S be a mod 2 relation with $w(S) \leq 12$. Suppose that S is stable under complex conjugation.*

(a) *There is a partition of S in which each part is either a conjugation-stable indecomposable relation or the disjoint union of two conjugate indecomposable relations.*
(b) *If S has even weight, then each conjugation-stable indecomposable relation in* (a) *has even weight.*

Proof. (a) We use induction on $w(S)$. Let $T \subset S$ be any indecomposable relation. Let T' be its conjugate. If $T = T'$ or $T \cap T' = \emptyset$, remove $T \cup T'$ from S and apply the inductive hypothesis. Otherwise, apply induction to $T \oplus T'$ and its complement in S.

(b) A conjugation-stable relation has odd weight if and only if it contains 1. $\square$

Theorem 6.10. *Let $x_1, \dots, x_n \in \mathbb{Q}$ be a sequence with $n \leq 6$ such that $\sum_{j=1}^{n} 2\cos(2\pi x_j) \equiv 0 \pmod{2\mathbb{Z}[\mu]}$, but no nonempty proper subsequence has the same property. Then either $n = 1$ and $2x_1 \in \mathbb{Z}$, or the given sequence can be obtained from one of the sequences listed in Table 2 by some combination of permutation of terms, individual negation, and* individual *translation by half-integers.*

Proof. This follows by applying Theorem 6.7 and Lemma 6.9 to the mod 2 cyclotomic relation coming from the sum $\sum_{j=1}^{n} \left(e^{2\pi i x_j} + e^{-2\pi i x_j}\right)$, except in the case where this sum cancels completely mod 2. Given the indecomposability hypothesis on the original sequence (which implies in particular that the x_j are distinct modulo $\frac{1}{2}\mathbb{Z}$), this happens only if $n = 1$ and $2x_1 \in \mathbb{Z}$. $\square$

Remark 6.11. Theorem 6.7 implies that for each $w \in \{3, \dots, 12\}$, reduction modulo 2 defines a weight-preserving bijection between equivalence classes of weight w indecomposable cyclotomic relations and equivalence classes of weight w indecomposable mod 2 cyclotomic relations, but this does not hold for all w. The cyclotomic polynomial Φ_{105} has two coefficients equal to -2; it yields a weight 35 indecomposable cyclotomic relation reducing to a weight 31 indecomposable mod 2 relation; see [DZ02, p. 105].

7. Torsion closures

Throughout this section, let K be a subfield of $\mathbb{C}$, let T be the torus $\mathbb{G}_m^n = \operatorname{Spec} K[x_1^{\pm}, \dots, x_n^{\pm}]$, and let X be a closed subscheme of T. For $P \in T(K)$, let $t_P \colon T \to T$ be the translation-by-P map. For a positive integer m, let $[m] \colon T \to T$ be the mth power map, and let $T[m] \subset T(\mathbb{C})$ be its kernel.

Definition 7.1. A *torsion coset* of T is a translate of a subtorus of $T_{\mathbb{C}}$ by a point in $\bigcup_{m \geq 1} T[m]$.

Definition 7.2. The *torsion closure* of X in T is the Zariski closure of $X(\mathbb{C}) \cap \mu^n$, viewed as a reduced K-subscheme of X.

Theorem 7.3 (Laurent)**.** *The torsion closure of $X_{\mathbb{C}}$ is a finite union of torsion cosets of T.*

Proof. This is a special case of [Lau84]. It is also a special case of a conjecture of Lang [Lan83, Notes for Chapter 8] combining the Manin–Mumford and Mordell conjectures, which is itself now known in full generality [McQ95]. See also [Hin06] for a survey. □

Since torsion cosets are definable over $\mathbb{Q}(\mu)$, the general problem of computing torsion closures can be reduced to the case in which X is defined over the field $K = \mathbb{Q}(\zeta_N)$ for some N; see [AS12, §3.3].

The key idea behind our algorithm for computing torsion closures is that certain field automorphisms act on torsion points in the same way as certain morphisms of varieties; for example, there is an automorphism of $\mathbb{C}$ that acts on odd-order roots of unity in the same way as the squaring morphism $\mathbb{G}_m \to \mathbb{G}_m$. This idea appears in the proof of the case $n = 2$ of Laurent's theorem by Ihara, Serre, and Tate [Lan83, §8.6], and in subsequent presentations by Ruppert [Rup93], Beukers and Smyth [BS02], and Aliev and Smyth [AS12].

In writing $K = \mathbb{Q}(\zeta_N)$, we may assume that $N = 2^e m$ with $e \geq 1$ and m odd. If $e = 1$, choose $\tau \in \operatorname{Aut} K$ such that $\tau(\zeta_m) = \zeta_m^2$. If $e \geq 2$, choose $\sigma \in \operatorname{Aut} K$ such that $\sigma(\zeta_N) = -\zeta_N$. Extend τ to a $\mathbb{Q}$-automorphism of $K[x_1^{\pm}, \ldots, x_n^{\pm}]$ acting trivially on the x_i. Then $\operatorname{Spec} \tau$ is a $\mathbb{Q}$-endomorphism of T. Similarly, define $\operatorname{Spec} \sigma \in \operatorname{Aut}_{\mathbb{Q}} T$. Define the following finite sets of $\mathbb{Q}$-endomorphisms of T:

$$\begin{aligned}
S_1 &:= \{t_P : P \in T[2] \setminus \{1\}\}, \\
S_2 &:= \begin{cases} \emptyset, & \text{if } e = 1, \\ \{(\operatorname{Spec} \sigma) \circ t_P : P \in T[2]\}, & \text{if } e \geq 2, \end{cases} \\
S_3 &:= \begin{cases} \{(\operatorname{Spec} \tau) \circ t_P \circ [2] : P \in T[2]\}, & \text{if } e = 1, \\ \emptyset, & \text{if } e \geq 2. \end{cases}
\end{aligned}$$

Let $S = S_1 \cup S_2 \cup S_3$. Intersections of subschemes below are always scheme-theoretic intersections.

Lemma 7.4. *The torsion closure of X is contained in $\bigcup_{f \in S}(X \cap f^{-1}X)$.*

Proof. Each $\alpha \in \operatorname{Aut} \mathbb{C}$ induces a coordinatewise map $T(\mathbb{C}) \to T(\mathbb{C})$. Suppose that $w \in T(\mathbb{C})$ and $\alpha(w) = z$. Then $\operatorname{Spec} \alpha|_K$ maps the image of $\operatorname{Spec} \mathbb{C} \xrightarrow{z} T$ to the image of $\operatorname{Spec} \mathbb{C} \xrightarrow{w} T$, because the diagram

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{C} \arrow[r, "\operatorname{Spec} \alpha"] \arrow[d, "z"'] & \operatorname{Spec} \mathbb{C} \arrow[d, "w"] \\
T \arrow[r, "\operatorname{Spec} \alpha|_K"] & T
\end{tikzcd}$$

commutes (check on rings); in particular, if $w \in X(\mathbb{C})$, then $z \in ((\operatorname{Spec} \alpha|_K)^{-1} X)(\mathbb{C})$.

Let $z = (z_1, \ldots, z_n) \in X(\mathbb{C})$ be a torsion point. Write $K(z) \cap \mu = \langle \zeta_M \rangle$, so $N | M$.

- Suppose that $4 \nmid M$. Extend τ to $\alpha \in \operatorname{Aut} \mathbb{C}$ such that $\alpha(\zeta_M) = -\zeta_M^2$. Then $\alpha(z_i) = \pm z_i^2$ for each i, so $\alpha(z) = (t_P \circ [2])(z)$ for some $P \in T[2]$. By the first paragraph of the proof, $(t_P \circ [2])(z) \in (\operatorname{Spec} \tau)^{-1} X)(\mathbb{C})$, so $z \in (f^{-1}X)(\mathbb{C})$ for some $f \in S_3$.
- Suppose that $4 | M$. Let $\sigma' \in \operatorname{Aut} K$ be 1 or σ, according to whether M/N is even or not. Then σ' extends to $\alpha \in \operatorname{Aut} \mathbb{C}$ such that $\alpha(\zeta_M) = -\zeta_M$. Then $\alpha(z_i) = \pm z_i$ for all i, so $\alpha(z) = t_P(z)$ for some $P \in T[2]$. If $P = 1$, then α fixes z but not $K(z)$, so $\sigma' = \alpha|_K \neq 1$. By the first paragraph again, $z \in (f^{-1}X)(\mathbb{C})$ for some f in S_1 or S_2, according to whether σ' is 1 or σ. □

Lemma 7.4 suggests the following recursive algorithm for computing the torsion closure of X.

Algorithm 7.5. Suppose that $K = \mathbb{Q}(\zeta_N)$. Given a closed subscheme X of T, return another closed subscheme of T as follows.

1. If $X \subseteq f^{-1}X$ for some $f \in S$, then choose one such f (using any deterministic tiebreaker) and proceed as follows.
   a. If $f \in S_1$, compute the closed subgroup $T_0 := \mathrm{Stab}_T(X)$ of T (see [Mil17, Corollary 1.81]), apply the algorithm to $X/T_0 \subset T/T_0$, and return the pullback of the result along $T \to T/T_0$.
   b. If $f \in S_2$, let $K' = \mathbb{Q}(\zeta_{N/2})$, write $P = ((-1)^{e_1}, \dots, (-1)^{e_n})$ with $e_i \in \{0,1\}$, and put $Q = (\zeta_N^{e_1}, \dots, \zeta_N^{e_n})$, so that $\sigma(Q)/Q = P$. Then $\operatorname{Spec}\sigma$ preserves $t_Q(X) \subset T$, and taking quotients (or invariant coordinate rings) yields $X' \subset T' = \mathbb{G}^n_{m,K'}$. Apply the algorithm to X' in T' over K', and return the pullback of the result along $T \xrightarrow{t_Q} T \to T'$.
   c. If $f \in S_3$, then check whether X is reducible. If so, return the union of the torsion closures of the irreducible components of X; otherwise, return the reduced subscheme of X.
2. If $X \not\subseteq f^{-1}X$ for every $f \in S$, then apply the algorithm to $X \cap f^{-1}X$ for each $f \in S$ and return the union of the results.

Theorem 7.6. *Algorithm 7.5 returns the torsion closure of X.*

Proof. We first verify termination. It suffices to check that no branch of the recursion can proceed to infinite depth. In step 1a, $f = t_P$ for some nontrivial P, and $P \in T_0$, so $T_0 \neq \{1\}$; thus step 1a cannot occur twice without an instance of step 2 in between. In step 1b, we replace K with a smaller number field; thus along a given branch, step 1b cannot occur more than $[K : \mathbb{Q}]$ times without an instance of step 2 in between. Finally, along each branch, after steps 1a and 1b occur for the last time, steps 1c and 2 can occur only finitely many times since T is noetherian.

We next verify correctness. The reduction in step 2 is valid by Lemma 7.4. The reductions in steps 1a and 1b are valid since torsion closures respect field extension and pullback by isogenies or translations by torsion points. The reduction in the reducible case of step 1c is valid since torsion closures can be computed on irreducible components. Finally, in the irreducible case of step 1c, the reduced subscheme of X equals the torsion closure, by the following lemma. $\square$

Lemma 7.7. *Let K be a number field. Fix $\tau \in \operatorname{Aut} K$, a torsion point $P \in T(K)$, and an integer $m \geq 2$. Let $f \colon T \to T$ be the $\mathbb{Q}$-morphism $x \mapsto (\operatorname{Spec}\tau) \circ t_P \circ [m]$. If X is integral and $f(X) \subseteq X$, then X equals its torsion closure.*

Proof. By replacing f by an iterate, we may assume that $\tau = 1$. Let $Z_1, \dots, Z_r$ be the irreducible components of $X_{\overline{K}}$. Since X is integral, the Z_i are Galois conjugates, so they have the same dimension. Since f is finite, it maps Z_1 to some Z_j, and then the conjugates of Z_1 are mapped to the conjugates of Z_j, so f induces a permutation of $\{Z_1, \dots, Z_r\}$. By replacing f by an iterate, we may assume that $f(Z_i) = Z_i$ for each i. By [Hin88, Lemme 10], Z_i is a translate of a subtorus of $T_{\overline{K}}$. Since $f(Z_i) = Z_i$, it must be a torsion coset. Thus X equals its torsion closure. $\square$

Remark 7.8. We have implemented a variant of Algorithm 7.5 in SageMath. To speed up the algorithm, we incorporated the following modifications:

- When we detect that the defining ideal of X contains a univariate polynomial, we factor this polynomial (after enlarging K suitably) so that we can reduce the dimension of T.
- When applying step 1a, we first check whether there exists a positive-dimensional subtorus T_1 of T such that X arises by pullback from T/T_1. If so, we use T_1 in place of T_0; otherwise, we use $\langle t_P \rangle$ in place of T_0. This does not affect termination.
- When applying step 2 to $f \in S_3$, at the next level of recursion we replace S by $\{f\}$. This does not affect correctness because this branch of the recursion needs to account for only torsion points of order not divisible by 4.
- We sometimes cut down X based on its reduction modulo 2, as in Section 6.

In experiments, we compute torsion closures easily when $\dim X \leq 1$, with difficulty when $\dim X = 2$, and not at all when $\dim X \geq 3$. Our use of Gröbner bases makes it difficult to analyze the running time, but a similar algorithm using resultants was analyzed in [AS12]; its complexity is superexponential in the number of variables.

8. Low order solutions to the Gram determinant equation

In this section, we prove the following statement.

Proposition 8.1. *For $N \in \{48, 90, 120, 132, 168, 280, 420\}$, every 4-line configuration with angles in $\mathbb{Z}\pi/N$ appears in some configuration accounted for in Theorem 1.2.*

To prove this, we make a rigorous computation of the solutions $\Theta = (\theta_{12}, \theta_{34}, \theta_{13}, \theta_{24}, \theta_{14}, \theta_{23}) \in \mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}}$ to (1) with $\theta_{ij} \in \mathbb{Z}\pi/N$. This computation combines numerical and algebraic methods. In Section 9, a separate computation will show that these account for all solutions outside of some specific families.

Write $\theta_{ij} = m_{ij}\pi/N$ with $m_{ij} \in \{1, \ldots, N-1\}$. By exploiting S_4-symmetry, we may assume

$$\begin{aligned} m_{14} + m_{23} \ &\leq \ m_{13} + m_{24} \ \leq \ m_{12} + m_{34}, \\ m_{34} \ &\leq \ m_{12}, \\ m_{24} \ &\leq \ m_{13}; \end{aligned} \tag{6}$$

but we cannot also assume $m_{23} \leq m_{14}$. The plan is to loop over $m_{12}, m_{34}, m_{13}, m_{24}, m_{14}$; then (1) expresses $\cos\theta_{23}$ as a root of a quadratic equation, so we can numerically solve for the possibilities for m_{23}, rounding them to the nearest integer, and carry out three tests on the resulting 6-tuple:

i) We test whether (1) holds to within 10^{-11} using `C++` and double precision arithmetic. (In a few cases, this requires computing m_{23} using more working precision; see below.)
ii) If the first test passes, we test whether (1) holds to within 10^{-50} using Bailey's `C++` quad-double package [QD] (approximately 65 decimal digits of working precision).
iii) If the second test passes, we rigorously verify (2) and hence (1) by an algebraic computation in $\mathbb{Q}(\zeta_{2N})$ in SageMath.

To save time, we precompute the values of $\cos(m\pi/N)$ and $\cos^2(m\pi/N)$, in both double and quad-double precision, for all $m < N$.

The first and second test were run on a MacBook Pro with a 2.9GHz Intel Core i7 CPU. The case of $N = 420$ dominated the time needed and took one day of computation using one core of the CPU. Most tuples were ruled out by the first test, so the second and third tests took a negligible amount of time.

While the third test, being algebraic, confirms rigorously that we have no false positives, we must do some analysis to rule out false negatives. We state this in the form of a lemma.

Lemma 8.2. *For $N \leq 420$, any tuple satisfying* (1) *and* (6) *passes tests 1 and 2.*

Proof. We begin with some observations about the accuracy of underlying floating-point arithmetic in our computations. Note that `C++` doubles correspond to IEEE-754 doubles, with 52 of 64 bits devoted to the mantissa, i.e., at least 15 decimal digits. Moreover, we compared the cosine values in doubles to quad doubles, finding agreement to within 10^{-15}; this ensures the accuracy of the cosines in doubles to more than 50 bits.

For a 6-tuple eliminated in the first test, we have to rule out a relative error of greater than 10^{-4} (since the cosines are accurate to 15 decimal places, whereas only 11 decimal places are used to distinguish the determinant (1) from zero). The computation of the determinant from the matrix entries involves a few dozen multiplications and additions of cosines that are bounded in size by 1, and thus is quite safe *provided* that m_{23} is correctly computed from the other values. That is, let

$\cos\theta_{23}$ be a root of $Ax^2 + Bx + C$ with $A = \cos^2\theta_{14} - 1$ and B and C given by more complicated expressions in the five other cosines; we must ensure that when we solve the quadratic equation to obtain

$$m_{23} = \frac{N}{\pi} \arccos \frac{-B \pm (B^2 - 4AC)^{1/2}}{2A}, \tag{7}$$

we are guaranteed to obtain m_{23} to within 0.5. In general this will not yield an integer, but we nonetheless round the computed value to the nearest integer and then test whether (1) holds. (In most cases this is redundant because the true value of m_{23} is not an integer, but this extra step takes negligible time due to our use of precomputed values, as described above.)

We analyze the numerical stability of (7) by stepping through the computation. The denominator, $2A = 2(\cos^2\theta_{14} - 1)$, can be as small as $2(\cos^2(\pi/420) - 1) = 0.00011\cdots$. Let $\alpha = -B \pm (B^2 - 4AC)^{1/2}$ and $\beta = 2A$, and let $\alpha + \Delta_1, \beta + \Delta_2$ be the numerical values computed for α, β; then

$$\frac{\alpha}{\beta} - \frac{\alpha + \Delta_1}{\beta + \Delta_2} = \frac{\alpha\Delta_2 - \beta\Delta_1}{\beta(\beta + \Delta_2)}.$$

The factor $\alpha\Delta_2/\beta^2$ can act to magnify the error; for $\alpha \approx 10$ and $\beta = 0.00011\cdots$, this is roughly 10^9. Additionally, taking the arccos introduces a further factor of $(N/\pi)(1 - x^2)^{-1/2}$ to the error, coming from the mean value theorem applied to $(N/\pi)\arccos(x)$; in the worst case $N = 420$ and $x \approx \pi/420$, this yields a factor of ≈ 18000. We conclude that in a few cases, we may lose more than 13 decimal digits of accuracy; however, if $(|\alpha| + |\beta|)/|\beta|^2 < 10^8$, the previous analysis guarantees that m_{23} is safely computed correctly using double precision. In the remaining cases, we recomputed m_{23} in quad-double precision to confirm its value; in practice we only had to resort to quad doubles for this step, in total, for less than 1/2000 of the cases examined, and this had a negligible impact on the overall runtime. ☐

After determining whether a given 6-tuple $(m_{12}, m_{34}, m_{13}, m_{24}, m_{14}, m_{23})$ is a solution to (1), we further check whether condition 1 in Proposition 2.2 holds. By Remark 2.4, this is the same as checking that the four 3×3 principal minors of the Gram matrix in (1) are nonnegative. To do so, we numerically compute the four minors using quad-double precision (65 digits precision), and declare each one to be nonnegative if its computed value is greater than -10^{-50}; the following lemma shows that this test is rigorous.

Lemma 8.3. *For a solution $(m_{12}, m_{34}, m_{13}, m_{24}, m_{14}, m_{23})$ of (1) with $0 < m_{ij} < N \leq 420$, if some 3×3 principal minor of the Gram matrix is nonzero, then its absolute value is greater than 10^{-50}.*

Proof. Without loss of generality, consider the top left 3×3 minor; it is $1 + 2\cos\theta_{12}\cos\theta_{13}\cos\theta_{23} - \cos^2\theta_{12} - \cos^2\theta_{13} - \cos^2\theta_{23}$, which equals

$$4 \sin\frac{\theta_{12} + \theta_{13} + \theta_{23}}{2} \sin\frac{\theta_{12} + \theta_{13} - \theta_{23}}{2} \sin\frac{\theta_{12} - \theta_{13} + \theta_{23}}{2} \sin\frac{-\theta_{12} + \theta_{13} + \theta_{23}}{2}. \tag{8}$$

One can verify this identity using trigonometric identities, or by writing $\cos(t)$ as $(z + 1/z)/2$ with $z = \exp(it)$ and factoring the corresponding Laurent polynomial, but it is suggested by noticing that the above expression vanishes when any of the inequalities in Remark 2.5 are equalities. For $N \leq 420$, each factor, if nonzero, has absolute value at least $\sin(\pi/(2N)) \geq 0.0074799\cdots$. This yields the desired bound by a wide margin (even 10^{-10} would suffice). ☐

Remark 8.4. Equation (8) also gives the square of the volume of a parallelepiped formed by three unit vectors in terms of the angles between them.

By Proposition 2.2, any solution whose Gram matrix has nonnegative 3×3 principal minors is realized by 4 vectors in $\mathbb{R}^3$. We then sort solutions according to whether the underlying 4 vectors are:

(i) all lying on one plane;
(ii) 3 vectors lying on a plane and one vector perpendicular to that plane;
(iii) 3 vectors lying on a plane with the origin in their convex hull, and the fourth vector neither on nor perpendicular to the plane;
(iv) outward normals to the faces of a tetrahedron;
(v) none of the above.

To test for condition (i), we check whether all four 3×3 principal minors are zero (implying that any three of the vectors are coplanar). Using Lemma 8.3 again, we can test this rigorously by computing in quad-double precision and verifying that their absolute values are less than 10^{-50}.

To test for conditions (ii) and (iii), we first verify that exactly one of the four 3×3 principal minors is zero (and the other three positive), again using Lemma 8.3; if so, then we must have 3 vectors lying on a plane and the fourth not lying on the plane. We then test for (ii) and (iii) respectively by checking whether the values of m_{ij} corresponding to angles including (respectively, not including) the fourth vector are all equal to $N/2$ (respectively, sum up to N).

To test for condition (iv), we first verify that all four 3×3 principal minors are positive, again using Lemma 8.3, to ensure that the 4 vectors are in linear general position. In this case, condition (iv) asserts that the unique (up to scalar combination) vanishing linear combination of the 4 vectors has coefficients all of the same sign; using Cramer's rule, we check this by computing the signs of the 3×3 nonprincipal minors. For this, we need an analogue of Lemma 8.3 to reduce to a computation in quad doubles.

Lemma 8.5. *For a solution $(m_{12}, m_{34}, m_{13}, m_{24}, m_{14}, m_{23})$ of (1) with $0 < m_{ij} < N \leq 420$, if all diagonal cofactors of the Gram matrix are positive, then every off-diagonal cofactor has absolute value greater than 10^{-50}.*

Proof. Consider four unit vectors in $\mathbb{R}^3$ given by Proposition 2.2, and form a 3×4 matrix B with these as column vectors. Let B_j denote the 3×3 matrix obtained from B by removing column j. Then the 3×3 submatrix of the Gram matrix obtained by removing row i and column j equals $B_i^T B_j$, so the corresponding cofactor equals $(-1)^{i+j} \det(B_i) \det(B_j)$. Up to sign, this is the geometric mean of two diagonal cofactors of the Gram matrix; we may thus deduce the claim directly from Lemma 8.3. □

With this classification in hand, we discard solutions of type (i) and (ii) as trivial cases. We further filter solutions of type (iii) and (iv) for solutions in a known parametric family. (We ignore solutions of type (v); any such solution arises from a solution of type (iii) or (iv) by negating one or more vectors.) The remaining solutions are all accounted for by Theorem 1.2, so the proof of Proposition 8.1 is complete.

Remark 8.6. Since it did not take much extra effort, we ran our code for all $N \leq 280$ and $N = 420$. The extra values of N provide a sanity check for the correctness of the implementation. As an additional sanity check, we use the unfiltered solutions of types (iii) and (iv) to experimentally find two-parameter solutions, and then one-parameter solutions not contained in a two-parameter solution. These agree with the solutions computed algebraically as described in Section 9. We do this by by looping over all triples of solutions to (1) found across several stretches of N (such as $N < 100$). Any such triple determines a plane in $\mathbb{R}^6$. We then select 5 random points on each plane and test whether equation (1) holds for all five points, to within 10^{-11}. If it does, we declare the three points to be part of a two-parameter family of solutions, and then confirm that the family matches one of the two-parameter families found algebraically (conversely, every two-parameter family found algebraically was confirmed in this fashion). Using exact rational arithmetic, we remove all solutions from our list that are on that plane. After exhausting all triples, we repeat the process with all remaining pairs of solutions to experimentally determine the one-parameter families of solutions, and verify that they match those found in Section 9. The remaining solutions are thus the sporadic solutions. The sporadic solutions of type (iv) are listed in Table 3.

9. The 4-line configurations

In this section, we prove the following result in the direction of Theorem 1.2, then use this to deduce Theorem 1.9.

Theorem 9.1. *Every rational-angle 4-line configuration, up to equivalence, is contained in one of the configurations indicated in Theorem 1.2.*

Our approach is to combine the computational results of Section 8 with a partial classification of solutions of (1), initially done modulo the symmetries identified in Sections 3 and 4. While in principle it is not necessary to rely on the exhaustive computations, doing so makes the computations far more efficient and the results less vulnerable to programming errors.

Definition 9.2. Let Λ^* be the kernel of the homomorphism $\tau \circ \exp\colon \mathbb{R}^6 \cong \mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}} \to T(\mathbb{C})$ from (3); this is a lattice containing $(2\pi\mathbb{Z})^6$ with index 8. Let G be the group of affine-linear transformations of $\mathrm{M}_4(\mathbb{R})_0^{\mathrm{sym}}$ generated by

- the translation action of Λ^*;
- the action of $\{\pm 1\}^6$ by multiplication on coordinates; and
- the action of S_4.

Let G' be the group generated by G and $\mathfrak{R}$; note that G' acts on $\mathcal{H}$. A calculation shows that there is a short exact sequence $1 \to \Lambda^* \to G' \to W(D_6) \to 1$.

Lemma 9.3. *Let $\Theta \in \mathcal{P} \cap \mathcal{H} \cap (\mathbb{Q}\pi)^6$ be a matrix corresponding to a configuration of four lines, exactly three of which are coplanar. Then at least one of the following conditions holds:*

- *Θ is G'-equivalent to a matrix corresponding to a perpendicular 4-line configuration (a configuration with one line perpendicular to the other three);*
- *Θ is G'-equivalent to a matrix of one of the forms*

$$(x, x, 2\pi/3, \pi - 2x, \pi/2, x),\ (x, 2x, 2\pi/3, \pi - 3x, 2\pi/3, x),\ (\pi/3, 2x, \pi/2, \pi - 3x, \pi + x, x) \qquad (9)$$

  for some $x \in \mathbb{Q}\pi$;
- *Θ has entries in $\mathbb{Z}\pi/N$ for some $N \in \{90, 120, 132, 168, 280, 420\}$.*

Proof. We first compute G-orbit representatives for the set of matrices Θ as above. Each G-orbit contains a matrix of the form $\angle(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ where $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are coplanar with 0 in their convex hull, so that

$$\theta_{23} + \theta_{24} + \theta_{34} \equiv 0 \pmod{2\pi}; \qquad (10)$$

it is thus equivalent to compute G_1-orbits of such matrices, where G_1 is the subgroup of G that preserves (10). Substituting $z_{23} = z_{24}^{-1} z_{34}^{-1}$ into (2) yields a square; taking a square root leads to

$$\begin{aligned} &\cos(\tfrac{\pi}{2} + \theta_{12} - \theta_{34}) + \cos(\tfrac{\pi}{2} + \theta_{13} - \theta_{24}) + \cos(\tfrac{\pi}{2} + \theta_{14} - \theta_{23}) \\ &+ \cos(\tfrac{\pi}{2} - \theta_{12} - \theta_{34}) + \cos(\tfrac{\pi}{2} - \theta_{13} - \theta_{24}) + \cos(\tfrac{\pi}{2} - \theta_{14} - \theta_{23}) = 0. \end{aligned} \qquad (11)$$

Theorem 5.1 implies that any solution of (11) is a specialization of a combination of indecomposable relations of one of the following forms in the notation of Table 2, up to the transformations in Theorem 5.1:

$$6, 5+1, \qquad (12)$$

$$2^* + 2^* + 2^*, 3^* + 3^*, 3^* + 2^* + 1, 6^*, 5^* + 1, 4 + 2^*, 3^* + 3, 3 + 2^* + 1; \qquad (13)$$

here (12) lists the possibilities with no free parameters. We omit forms including $1 + 1$ because such a pair is a specialization of a relation of type 2^*. Similarly, we omit $3 + 3$.

Table 2 shows that any solution of (11) of a form in (12) has values in $\mathbb{Z}\pi/N$ for some $N \in \{132, 168, 280, 420\}$. Thus it remains to identify solutions of the equations (10) and (11) corresponding to forms listed in (13), modulo the action of G_1. The solutions will fall into finitely many families, each represented as the set of solutions to a system of congruences $A\Theta = 2\pi\mathbf{b} \pmod{2\pi}$ for some integer matrix A and rational vector $\mathbf{b}$. To put a collection of augmented matrices $(A|\mathbf{b})$ into a standard form, we perform the following operations until they have no further effect.

- Perform row reduction on each $(A|\mathbf{b})$ to put A into Hermite normal form, omitting zero rows.
- If a row $(a_1 \cdots a_6|b)$ of some $(A|\mathbf{b})$ has $d := \gcd(a_1, \ldots, a_6) > 1$, replace $(A|\mathbf{b})$ with the d matrices obtained by replacing this row in turn by $(\frac{a_1}{d} \cdots \frac{a_6}{d}|\frac{b+i}{d})$ for $i = 0, \ldots, d-1$.
- Reduce the coordinates of each $\mathbf{b}$ modulo 1 to put them in $[0, 1)$.

To intersect two families $(A_1|\mathbf{b}_1)$ and $(A_2|\mathbf{b}_2)$, perform the first operation on $\left(\begin{array}{c|c} A_1 & \mathbf{b}_1 \\ A_2 & \mathbf{b}_2 \end{array}\right)$. To test whether one family is contained in another, compare it to their intersection.

For each form in (13) and cosine relation of that form, each possible matching of this relation to the six angles

$$\tfrac{\pi}{2} \pm \theta_{12} - \theta_{34}, \qquad \tfrac{\pi}{2} \pm \theta_{14} - \theta_{23}, \qquad \tfrac{\pi}{2} \pm \theta_{14} - \theta_{23},$$

defines an augmented matrix $(A|\mathbf{b})$ as above. We put these matrices into standard form, eliminate any family contained in another family, and eliminate any family contained in one of the degenerate families

$$\theta_{jk} \equiv 0, \pi \pmod{2\pi}$$
$$\theta_{1j} \pm \theta_{1k} \pm \theta_{jk} \equiv 0 \pmod{2\pi};$$

the latter corresponds to $\mathbf{v}_1, \mathbf{v}_j, \mathbf{v}_k$ being coplanar in addition to $\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, making all four coplanar. These computations take about 2 hours in SageMath on a virtual 2.3GHz Intel Xeon CPU.

For each matrix $(A|\mathbf{b})$ in the result of the computation, we may solve the equation $A\mathbf{v} = \mathbf{b}$ to obtain an affine subspace of $\mathcal{H}$. Inspecting the output, we find the following.

- We obtain no subspaces of dimension three or more.
- We obtain 3 two-dimensional subspaces, which belong to a single G'-orbit. This orbit contains a subspace consisting entirely of perpendicular 4-line configurations.
- We obtain 13 one-dimensional subspaces, which belong to 3 distinct G'-orbits. These orbits are represented by the three subspaces listed in (9).
- The remaining subspaces are isolated points with coordinates in $\mathbb{Z}\pi/N$ for some $N \in \{84, 90, 120\}$. Any Θ that is G-equivalent to one of these also has coordinates in $\mathbb{Z}\pi/N$. □

Remark 9.4. The equations (10) and (11) constitute the same system as the one solved in [PR98, Theorem 4.4] to classify concurrent diagonals of regular polygons, except for some positivity conditions in the latter statement. Unfortunately, these conditions prevent us from deriving Lemma 9.3 directly from results in [PR98].

Lemma 9.5. *Let $\Theta \in \mathcal{P} \cap \mathcal{H} \cap (\mathbb{Q}\pi)^6$ be a matrix corresponding to a configuration of four lines, no three of which are coplanar. Then at least one of the following conditions holds:*

- *Θ is G'-equivalent to a matrix of the form*
$$(\pi/2, \pi/2, \pi - 2x, \pi/3, x, x) \tag{14}$$
for some $x \in \mathbb{Q}\pi$;
- *Θ has entries in $\mathbb{Z}\pi/N$ for some $N \in \{48, 120, 132, 168, 280, 420\}$.*

Proof. Multiplying the matrix in (1) by 2 and reducing modulo $2\mathbb{Z}[\mu]$ yields the congruence

$$\begin{aligned} &2\cos(2\theta_{12}+2\theta_{34})+2\cos(2\theta_{13}+2\theta_{24})+2\cos(2\theta_{14}+2\theta_{24}) \\ &+2\cos(2\theta_{12}-2\theta_{34})+2\cos(2\theta_{13}-2\theta_{24})+2\cos(2\theta_{14}-2\theta_{24})\equiv 0 \pmod{2\mathbb{Z}[\mu]}. \end{aligned} \tag{15}$$

Theorem 6.10 classifies solutions to (15) in terms of the indecomposable mod 2 relations listed in Table 2, which include a length 1 relation which we call $1a$, and the additional length 1 relation $2\cos 0 \equiv 0$ in Theorem 6.10, which we call $1b$; let 1 denote a relation of type $1a$ or $1b$. Explicitly, any solution is a specialization of a sum of relations of the following forms, up to the transformations in Theorem 6.10:

$$6,\ 5+1,\ 4+1a+1b; \tag{16}$$

$$2^*+2^*+2^*,\ 3^*+3^*,\ 3^*+2^*+1,\ 2^*+2^*+1a+1b,\ 6^*,\ 5^*+1,\ 4+2^*,\ 3^*+3,\ 3+2^*+1; \tag{17}$$

here (16) lists the possibilities with no free parameters. We omit forms including $1a+1a$ or $1b+1b$ because such pairs are specializations of a relation of type 2^*. Similarly, we omit $3+3$.

Table 2 shows that any solution of (15) of a type in (16) has values in $\mathbb{Z}\pi/N$ for some $N \in \{120, 132, 168, 280, 420\}$. It thus remains to identify solutions of (1) arising from solutions of (15) of forms listed in (17). For each such form, each mod 2 cosine relation of that form, and each possible matching of this relation to the six angles

$$2\theta_{12}\pm 2\theta_{34}, \qquad 2\theta_{13}\pm 2\theta_{24}, \qquad 2\theta_{14}\pm 2\theta_{23},$$

we obtain an ideal in the Laurent polynomial ring $\mathbb{Q}[z_{ij}^{\pm}]$ via the substitution $z_{jk}=e^{\mathrm{i}\theta_{jk}}$. At this point we have the option to replace this set of ideals with a set of G'-orbit representatives; this turns out to be worthwhile for forms containing two or more free parameters. We then impose the condition (1) by adding the generator (2) to each ideal.

Using the implementation in Remark 7.8, we compute the torsion closures of the corresponding varieties. Once this is done, we eliminate solutions which are degenerate because one of the angles equals 0 or π (corresponding to two of the lines coinciding) or because one of the 3×3 minors of the Gram matrix vanishes (corresponding to the four lines not being in linear general position). These computations take about 2 hours in SageMath on a virtual 2.3GHz Intel Xeon CPU.

Each irreducible component of each torsion closure in the output corresponds to a subset of $\mathcal{H}$ consisting of the $(2\pi\mathbb{Z})^6$-translates of some affine subspace of $\mathcal{H}$. Inspecting the output, we find the following.

- We obtain no subspaces of dimension two or more.
- The one-dimensional subspaces belong to the G'-orbit of the subspace listed in (14).
- The remaining subspaces are isolated points with coordinates in $\mathbb{Z}\pi/N$ for $N \in \{21, 24, 60\}$. Any element of the G'-orbit of one of these points has coordinates in $\mathbb{Z}\pi/(2N)$. □

For each Θ in Lemmas 9.3 and 9.5, we now determine which elements in its G'-orbit lie in $\mathcal{P}$.

Lemma 9.6. *The G'-orbit of each subspace in* (9) *or* (14) *yields a single $\mathfrak{R}^{\pm}$-orbit of one-parameter families of $\mathbb{R}^3$-realizable 4×4 rational-angle matrices. Each orbit is represented by a family yielding a 4-line subconfiguration of a 6-line configuration in Example 10.6 or 10.5, respectively.*

Proof. We have $G' = \mathfrak{R}^{\pm}\cdot(2\pi\mathbb{Z})^6\cdot\{\pm 1\}^6$; for example, the translation in Λ^* sending θ_{1j} to $\pi+\theta_{1j}$ for $j=2,3,4$ is the composition of an element of $\{\pm1\}^6$ with the element of $\mathfrak{R}^{\pm}$ sending θ_{1j} to $\pi-\theta_{1j}$ for $j=2,3,4$. Also, $\mathfrak{R}^{\pm}$ preserves $\mathcal{P}$. Therefore it suffices to determine, for each subspace V in the $\{\pm1\}^6$-orbit of a subspace in (9) or (14), which $\mathbf{a}\in(2\pi\mathbb{Z})^6$ are such that $\mathbf{a}+V$ intersects $\mathcal{P}$. Each V has a parametrization $\theta(t)=t\mathbf{v}+\mathbf{w}$ with $\mathbf{v}\in\mathbb{Z}^6$; then $\theta(t+2\pi)\equiv\theta(t) \pmod{(2\pi\mathbb{Z})^6}$, so it suffices to consider the finitely many $\mathbf{a}$ such that $\mathbf{a}+\theta([-\pi,\pi])$ intersects $[0,\pi]^6\supset\mathcal{P}$. We keep each such $\mathbf{a}+V$ for which $(\mathbf{a}+V)\cap\mathcal{P}\neq\emptyset$. Finally, we compute the $\mathfrak{R}^{\pm}$-orbit representatives. This computation takes about 25 minutes in SageMath on a 2.3GHz Intel Core i5 CPU. □

Proof of Theorem 9.1. Let $\mathcal{F}$ be a two-parameter family of matrices associated to the perpendicular 4-line configurations. By Lemmas 9.3 and 9.5, any family of realizable 4×4 rational-angle matrices with two or more parameters is in the $\mathfrak{R}^{\pm}$-orbit of $\mathcal{F}$. Operating by coset representatives of $S_4^{\pm} \subseteq \mathfrak{R}^{\pm}$ shows that this $\mathfrak{R}^{\pm}$-orbit is a union of three $S_4^{\pm}$-orbits; one is that of the (nonmaximal) family $\mathcal{F}$, and the other two are the maximal two-parameter families in Theorem 1.2. By Lemmas 9.3 and 9.5 combined with Lemma 9.6, every one-parameter family is in the $\mathfrak{R}^{\pm}$-orbit of one of four families; again we split each $\mathfrak{R}^{\pm}$-orbit into $S_4^{\pm}$-orbits and find that they are as described in Theorem 1.2. Finally, by Lemmas 9.3 and 9.5 combined with Proposition 8.1, the isolated configurations are as described by Theorem 1.2. $\square$

Proof of Theorem 1.9. For this, we do not need the configurations described in Lemma 9.3. We need only the $\mathfrak{R}^{\pm}$-orbit in Lemma 9.6 from (14) and the isolated configurations of Proposition 8.1. We compute representatives for the S_4-orbits and filter by the "test for condition (iv)" in the paragraph before Lemma 8.5 (for the one-parameter families, it turns out that the parameter range corresponding to configurations with no three lines coplanar is an open interval, so it suffices to check one interior sample point, by continuity of the signs of the minors). The result is that the $\mathfrak{R}^{\pm}$-orbit from (14) yields the two infinite families in Theorem 1.9, and the isolated configurations outside those yield the list in Table 3. $\square$

10. From 4-line configurations to n-line configurations

We now complete the proof of Theorem 1.2, by assembling rational-angle n-line configurations from the classification of 4-line configurations given by Theorem 9.1. This requires some care to make the computation feasible.

Proposition 10.1. *Let $n \geq 4$. Let $\mathcal{L}$ be an n-line configuration in $\mathbb{R}^3$. Then $\mathcal{L}$ is contained in a perpendicular configuration if and only if each 4-line subconfiguration of $\mathcal{L}$ is contained in a perpendicular configuration.*

Proof. Suppose that each 4-line subconfiguration is contained in a perpendicular configuration. Then for every four lines in $\mathcal{L}$, there is a unique plane containing at least three of them, and the fourth is either in the plane or perpendicular to it. Fix $L_1, L_2, L_3 \in \mathcal{L}$ lying in a plane P. Then the unique plane for $\{L_1, L_2, L_3, L_i\}$ for any other $L_i \in \mathcal{L}$ must be P, and L_i is either in P or is the line perpendicular to P. Thus $\mathcal{L}$ is contained in a perpendicular configuration. $\square$

For each $n \geq 4$, let $\mathscr{M}_n$ be the set of $\mathbb{R}^3$-realizable $n \times n$ rational-angle matrices.

Proposition 10.2. *For each $n \geq 4$, there exists a finite set $\mathscr{A}_n$ of affine $\mathbb{Q}$-subspaces of $\mathrm{M}_n(\mathbb{Q}\pi)_0^{\mathrm{sym}}$ such that $\mathscr{M}_n = \bigcup_{A \in \mathscr{A}_n} (A \cap \mathcal{P}_n)$.*

Proof. The calculations described in the preceding sections construct such a set $\mathscr{A}_n$ when $n = 4$.

Now suppose that $n > 4$. Let $\binom{[n]}{4}$ be the set of 4-element subsets of $\{1, \ldots, n\}$. For $I \in \binom{[n]}{4}$, let $p_I \colon \mathrm{M}_n(\mathbb{Q})_0^{\mathrm{sym}} \to \mathrm{M}_4(\mathbb{Q})_0^{\mathrm{sym}}$ be the projection giving the principal submatrix indexed by I. By Corollary 2.3, for each $n > 4$, a matrix $\Theta \in \mathrm{M}_n(\mathbb{R})$ is in $\mathscr{M}_n$ if and only if its 4×4 principal submatrices are in $\mathscr{M}_4$. Thus we may take $\mathscr{A}_n$ to be the set of nonempty intersections of the form $\bigcap_I p_I^{-1}(A_I)$ where each A_I ranges over $\mathscr{A}_4$ independently. $\square$

We may assume that each $A \in \mathscr{A}_n$ equals the affine span of $A \cap \mathcal{P}_n$. We may also assume that $\mathscr{A}_n$ is irredundant in the sense that if $A, A' \in \mathscr{A}_n$ satisfy $A \subset A'$, then $A = A'$. These conditions specify $\mathscr{A}_n$ uniquely.

Let $\mathscr{A}'_n$ be the set of $A \in \mathscr{A}_n$ such that $A \cap \mathcal{P}_n$ contains a matrix with no off-diagonal entries equal to 0 or π. Let $\mathscr{A}''_n$ be the set of $A \in \mathscr{A}'_n$ such that $A \cap \mathcal{P}_n$ contains a matrix corresponding to a line configuration not contained in a perpendicular configuration (for each n, this condition removes exactly $n+1$ elements: the one parametrizing planar configurations, and, for each i, the one parametrizing configurations with the ith vector perpendicular to all the others). The group $S_n^{\pm}$ acts on $\mathscr{A}_n$, $\mathscr{A}'_n$, and $\mathscr{A}''_n$. To prove Theorem 1.2, we need to compute the set $\mathscr{R}''_n$ of $S_n^{\pm}$-orbit representatives in $\mathscr{A}''_n$ for n up to 16 and verify that $\mathscr{R}''_{16} = \emptyset$ (we need 16, and not just 15, to rule out adding a 16th line to the icosidodecahedral configuration). By Proposition 10.1, we need only consider angle matrices whose upper left 4×4 submatrix is in $\mathscr{R}''_4$.

What complicates our task is that $\#\mathscr{A}'_4 = 84696$. It is not practical to loop over all $84696^{\binom{n}{4}}$ tuples (A_I) as suggested by the proof of Proposition 10.2, even for $n = 5$, let alone $n = 16$.

One could imagine computing a set $\mathscr{R}''_n$ by induction, using projections onto $(n-1) \times (n-1)$ principal submatrices instead of 4×4 principal submatrices. Suppose that $\mathscr{R}''_{n-1}$ is known. The action of $S_{n-1}^{\pm} \subset S_n^{\pm}$ shows that each $S_n^{\pm}$-orbit in $\mathscr{A}''_n$ has a representative with upper left $(n-1) \times (n-1)$ submatrix in $\mathscr{R}''_{n-1}$, but we do not have the freedom to assume simultaneously that the other $(n-1) \times (n-1)$ principal submatrices are in $\mathscr{R}''_{n-1}$; all we can assume is that they are in $\mathscr{A}'_{n-1}$. This is a problem, since $\mathscr{A}'_{n-1}$ for some of the larger values of n is much larger even than $\mathscr{A}'_4$ because even a single $S_{n-1}^{\pm}$-orbit can be huge.

Therefore instead we employ the following "early abort" inductive strategy. Start with the list $\mathscr{R}''_{n-1}$ of affine subspaces giving possibilities for the upper left $(n-1) \times (n-1)$ submatrix. In the first stage, fix $I \in \binom{[n]}{4} - \binom{[n-1]}{4}$ and try to reconcile each possibility for the upper left $(n-1) \times (n-1)$ submatrix with each possibility in $\mathscr{A}'_4$ for the $I \times I$ principal submatrix — this amounts to intersecting preimages of affine subspaces, as in the proof of Proposition 10.2. Most of these preimage intersections will be empty or will correspond to a family whose general member has an off-diagonal entry equal to 0 or π, so they need not be considered further; later on in the process we will also have intersections that are reduced to a point, and we can discard those too if the point happens not to satisfy the inequalities defining $\mathcal{P}_n$. In the second stage, choose a different $I' \in \binom{[n]}{4} - \binom{[n-1]}{4}$ and try to reconcile the undiscarded possibilities with the possibilities for the $I' \times I'$ principal submatrix by computing preimage intersections again. There are 84696 branches at each stage, but most of the branches abort immediately, and it turns out that the list of possibilities remains under control. After completing a stage for every subset in $\binom{[n]}{4} - \binom{[n-1]}{4}$, we have a list of affine subspaces whose $S_n^{\pm}$-orbits include all the subspaces in $\mathscr{A}''_n$. We then compute a distinguished representative of the $S_n^{\pm}$-orbit of each subspace and eliminate redundancies, to obtain $\mathscr{R}''_n$.

Remark 10.3. To save more time, one can totally order the set of $S_4^{\pm}$-orbits in $\mathscr{A}'_4$, with the ones represented by elements of $\mathscr{R}''_4$ coming first. This induces a pre-order on $\mathscr{A}'_4$ itself. Then, by acting by $S_n^{\pm}$, we may assume that for each subspace in $\mathscr{R}''_n$, obtained as the intersection of preimages of A_I, the subspace A_I for $I = \{1,2,3,4\}$ is less than or equal to the A_J for every other $J \in \binom{[n]}{4}$. Thus when seeding the inductive process with a particular $A_{\{1,2,3,4\}}$, we need only consider A_J that are greater than equal to that one in each stage. By choosing the total ordering judiciously, starting with affine subspaces corresponding to line configurations that are unlikely to extend much, we greatly reduce the number of branches in stages for larger n. In fact, for simplicity we use a total pre-order instead of a total order; in other words, we group the orbits into clumps, and totally order the clumps. These improvements reduce the running time of all the calculations in this section to a total of 14 hours in Magma on a 3.5GHz Intel Xeon CPU E5-1620 v3.

Example 10.4. Each of the five 8-line configurations consists of seven of the central diagonals of a 60-gon centered at $\mathbf{0}$ together with one line neither in its plane nor perpendicular to it. Each of the five configurations has a different angle set, though in each case the angles are among the multiples of $\pi/30$.

Example 10.5. One of the one-parameter families of 6-line configurations is obtained by taking the lines spanned by $(1,0,0)$ and $\left(0, -\frac{2}{\sqrt{3}}\cos\theta, \sqrt{1-\frac{4}{3}\cos^2\theta}\right)$ and their rotations by $\pm 2\pi/3$ about the z-axis, for each parameter value $\theta \in \mathbb{Q}\pi \cap (\pi/6, \pi/2)$. The angles formed are $\pi/2$, $2\pi/3$, θ, $\pi - \theta$, and $\pi - 2\theta$.

Example 10.6. Three more one-parameter families of 6-line configurations can be obtained by taking the lines spanned by $(\cos r\theta, \sin r\theta, 0)$ for $r \in \{-2,-1,0,1,2\}$ and $\left(0, \frac{1}{2}\csc\theta, \sqrt{1-\frac{1}{4}\csc^2\theta}\right)$, for $\theta \in \mathbb{Q}\pi$ in one of the parameter ranges $(\pi/6, \pi/4)$, $(\pi/4, \pi/3)$, or $(\pi/3, \pi/2)$. The angles formed are $\pi/3$, $\pi/2$, $2\pi/3$, θ, 2θ, 3θ, and 4θ (apply $x \mapsto 2\pi - x$ if they exceed π).

11. Tables

We tabulate our results in a somewhat compressed form. A more verbose description can be found in the GitHub repository mentioned near the end of Section 1.

11.1. **Sporadic tetrahedra.** Table 3 lists the 59 similarity classes of tetrahedra with rational dihedral angles not belonging to one of the two parametric families described in Theorem 1.9. Each entry in the table lists the dihedral angles $(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23})$ measured in units of π/N for the integer N listed in the left column. The horizontal lines indicate groupings into orbits for the group $\mathfrak{R}$ generated by Regge symmetries (see Section 4). For those tetrahedra listed in [Bol78, pp. 170–173], we have included the labels used therein; all of these correspond to rational-angle 4-line configurations contained in either the 9-line or the 15-line maximal configuration.

N	$(\alpha_{12}, \alpha_{34}, \alpha_{13}, \alpha_{24}, \alpha_{14}, \alpha_{23})$ as multiples of π/N
12	$(3, 4, 3, 4, 6, 8) = H_2(\pi/4)$
24	$(5, 9, 6, 8, 13, 15)$
12	$(3, 6, 4, 6, 4, 6) = T_0$
24	$(7, 11, 7, 13, 8, 12)$
15	$(3, 3, 3, 5, 10, 10) = T_{18}$, $(2, 4, 4, 4, 10, 10)$, $(3, 3, 4, 4, 9, 11)$
15	$(3, 3, 5, 5, 9, 9) = T_7$
15	$(5, 5, 5, 9, 6, 6) = T_{23}$, $(3, 7, 6, 6, 7, 7)$, $(4, 8, 5, 5, 7, 7)$
21	$(3, 9, 7, 7, 12, 12)$, $(4, 10, 6, 6, 12, 12)$, $(6, 6, 7, 7, 9, 15)$
30	$(6, 12, 10, 15, 10, 20) = T_{17}$, $(4, 14, 10, 15, 12, 18)$
60	$(8, 28, 19, 31, 25, 35)$, $(12, 24, 15, 35, 25, 35)$,
	$(13, 23, 15, 35, 24, 36)$, $(13, 23, 19, 31, 20, 40)$
30	$(6, 18, 10, 10, 15, 15) = T_{13}$, $(4, 16, 12, 12, 15, 15)$, $(9, 21, 10, 10, 12, 12)$
30	$(6, 6, 10, 12, 15, 20) = T_{16}$, $(5, 7, 11, 11, 15, 20)$
60	$(7, 17, 20, 24, 35, 35)$, $(7, 17, 22, 22, 33, 37)$,
	$(10, 14, 17, 27, 35, 35)$, $(12, 12, 17, 27, 33, 37)$
30	$(6, 10, 10, 15, 12, 18) = T_{21}$, $(5, 11, 10, 15, 13, 17)$
60	$(10, 22, 21, 29, 25, 35)$, $(11, 21, 19, 31, 26, 34)$,
	$(11, 21, 21, 29, 24, 36)$, $(12, 20, 19, 31, 25, 35)$
30	$(6, 10, 6, 10, 15, 24) = T_6$
60	$(7, 25, 12, 20, 35, 43)$
30	$(6, 12, 6, 12, 15, 20) = T_2$
60	$(12, 24, 13, 23, 29, 41)$
30	$(6, 12, 10, 10, 15, 18) = T_3$, $(7, 13, 9, 9, 15, 18)$
60	$(12, 24, 17, 23, 33, 33)$, $(14, 26, 15, 21, 33, 33)$,
	$(15, 21, 20, 20, 27, 39)$, $(17, 23, 18, 18, 27, 39)$
30	$(6, 15, 6, 18, 10, 20) = T_4$, $(6, 15, 7, 17, 9, 21)$
60	$(9, 33, 14, 34, 21, 39)$, $(9, 33, 15, 33, 20, 40)$,
	$(11, 31, 12, 36, 21, 39)$, $(11, 31, 15, 33, 18, 42)$
30	$(6, 15, 10, 15, 12, 15) = T_1$, $(6, 15, 11, 14, 11, 16)$, $(8, 13, 8, 17, 12, 15)$,
	$(8, 13, 9, 18, 11, 14)$, $(8, 17, 9, 12, 11, 16)$, $(9, 12, 9, 18, 10, 15)$
30	$(10, 12, 10, 12, 15, 12) = T_5$
60	$(19, 25, 20, 24, 29, 25)$

TABLE 3. The 59 sporadic rational tetrahedra.

11.2. **Maximal rational-angle n-line configurations for $5 \le n \le 15$.** Each entry in the following list is a representative of an $S_n^{\pm}$-orbit of $\mathbb{R}^3$-realizable $n \times n$ rational-angle matrices for some $n \geq 5$, with each angle measured in units of π. The list is complete except that we omit families whose general member is contained in such an $N \times N$ matrix for some $N > n$.

$$
\begin{pmatrix}
0 & 1/5 & 1/5 & 1/5 & 1/5 & 1/3 & 1/3 & 1/3 & 1/3 & 2/5 & 2/5 & 2/5 & 2/5 & 1/2 & 1/2 \\
1/5 & 0 & 1/5 & 1/3 & 2/5 & 1/5 & 1/3 & 2/5 & 1/2 & 1/5 & 1/3 & 1/2 & 3/5 & 1/3 & 2/5 \\
1/5 & 1/5 & 0 & 2/5 & 1/3 & 2/5 & 1/2 & 1/5 & 1/3 & 1/3 & 1/5 & 3/5 & 1/2 & 1/3 & 3/5 \\
1/5 & 1/3 & 2/5 & 0 & 1/5 & 1/3 & 1/5 & 1/2 & 2/5 & 1/2 & 3/5 & 1/5 & 1/3 & 2/3 & 2/5 \\
1/5 & 2/5 & 1/3 & 1/5 & 0 & 1/2 & 2/5 & 1/3 & 1/5 & 3/5 & 1/2 & 1/3 & 1/5 & 2/3 & 3/5 \\
1/3 & 1/5 & 2/5 & 1/3 & 1/2 & 0 & 1/5 & 3/5 & 2/3 & 1/5 & 1/2 & 2/5 & 2/3 & 2/5 & 1/5 \\
1/3 & 1/3 & 1/2 & 1/5 & 2/5 & 1/5 & 0 & 2/3 & 3/5 & 2/5 & 2/3 & 1/5 & 1/2 & 3/5 & 1/5 \\
1/3 & 2/5 & 1/5 & 1/2 & 1/3 & 3/5 & 2/3 & 0 & 1/5 & 1/2 & 1/5 & 2/3 & 2/5 & 2/5 & 4/5 \\
1/3 & 1/2 & 1/3 & 2/5 & 1/5 & 2/3 & 3/5 & 1/5 & 0 & 2/3 & 2/5 & 1/2 & 1/5 & 3/5 & 4/5 \\
2/5 & 1/5 & 1/3 & 1/2 & 3/5 & 1/5 & 2/5 & 1/2 & 2/3 & 0 & 1/3 & 3/5 & 4/5 & 1/5 & 1/3 \\
2/5 & 1/3 & 1/5 & 3/5 & 1/2 & 1/2 & 2/3 & 1/5 & 2/5 & 1/3 & 0 & 4/5 & 3/5 & 1/5 & 2/3 \\
2/5 & 1/2 & 3/5 & 1/5 & 1/3 & 2/5 & 1/5 & 2/3 & 1/2 & 3/5 & 4/5 & 0 & 1/3 & 4/5 & 1/3 \\
2/5 & 3/5 & 1/2 & 1/3 & 1/5 & 2/3 & 1/2 & 2/5 & 1/5 & 4/5 & 3/5 & 1/3 & 0 & 4/5 & 2/3 \\
1/2 & 1/3 & 1/3 & 2/3 & 2/3 & 2/5 & 3/5 & 2/5 & 3/5 & 1/5 & 1/5 & 4/5 & 4/5 & 0 & 1/2 \\
1/2 & 2/5 & 3/5 & 2/5 & 3/5 & 1/5 & 1/5 & 4/5 & 4/5 & 1/3 & 2/3 & 1/3 & 2/3 & 1/2 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 1/4 & 1/4 & 1/4 & 1/4 & 1/2 & 1/2 & 1/2 & 1/2 \\
1/4 & 0 & 1/3 & 1/3 & 1/2 & 1/4 & 1/3 & 1/3 & 1/2 \\
1/4 & 1/3 & 0 & 1/2 & 1/3 & 1/2 & 1/3 & 2/3 & 1/4 \\
1/4 & 1/3 & 1/2 & 0 & 1/3 & 1/2 & 2/3 & 1/3 & 3/4 \\
1/4 & 1/2 & 1/3 & 1/3 & 0 & 3/4 & 2/3 & 2/3 & 1/2 \\
1/2 & 1/4 & 1/2 & 1/2 & 3/4 & 0 & 1/4 & 1/4 & 1/2 \\
1/2 & 1/3 & 1/3 & 2/3 & 2/3 & 1/4 & 0 & 1/2 & 1/4 \\
1/2 & 1/3 & 2/3 & 1/3 & 2/3 & 1/4 & 1/2 & 0 & 3/4 \\
1/2 & 1/2 & 1/4 & 3/4 & 1/2 & 1/2 & 1/4 & 3/4 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 1/10 & 1/6 & 4/15 & 3/10 & 2/5 & 13/30 & 7/15 \\
1/10 & 0 & 4/15 & 1/6 & 2/5 & 13/30 & 1/3 & 17/30 \\
1/6 & 4/15 & 0 & 13/30 & 2/15 & 11/30 & 3/5 & 3/10 \\
4/15 & 1/6 & 13/30 & 0 & 17/30 & 1/2 & 1/6 & 11/15 \\
3/10 & 2/5 & 2/15 & 17/30 & 0 & 11/30 & 11/15 & 1/6 \\
2/5 & 13/30 & 11/30 & 1/2 & 11/30 & 0 & 17/30 & 2/5 \\
13/30 & 1/3 & 3/5 & 1/6 & 11/15 & 17/30 & 0 & 9/10 \\
7/15 & 17/30 & 3/10 & 11/15 & 1/6 & 2/5 & 9/10 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 1/15 & 1/15 & 3/10 & 3/10 & 7/15 & 7/15 & 1/2 \\
1/15 & 0 & 2/15 & 7/30 & 11/30 & 2/5 & 8/15 & 7/15 \\
1/15 & 2/15 & 0 & 11/30 & 7/30 & 8/15 & 2/5 & 8/15 \\
3/10 & 7/30 & 11/30 & 0 & 3/5 & 1/6 & 23/30 & 11/30 \\
3/10 & 11/30 & 7/30 & 3/5 & 0 & 23/30 & 1/6 & 19/30 \\
7/15 & 2/5 & 8/15 & 1/6 & 23/30 & 0 & 14/15 & 1/3 \\
7/15 & 8/15 & 2/5 & 23/30 & 1/6 & 14/15 & 0 & 2/3 \\
1/2 & 7/15 & 8/15 & 11/30 & 19/30 & 1/3 & 2/3 & 0
\end{pmatrix}
$$

$$\begin{pmatrix} 0 & 1/15 & 1/5 & 4/15 & 4/15 & 11/30 & 13/30 & 7/15 \\ 1/15 & 0 & 4/15 & 1/5 & 4/15 & 13/30 & 11/30 & 8/15 \\ 1/5 & 4/15 & 0 & 7/15 & 1/3 & 1/6 & 19/30 & 4/15 \\ 4/15 & 1/5 & 7/15 & 0 & 1/3 & 19/30 & 1/6 & 11/15 \\ 4/15 & 4/15 & 1/3 & 1/3 & 0 & 13/30 & 13/30 & 1/2 \\ 11/30 & 13/30 & 1/6 & 19/30 & 13/30 & 0 & 4/5 & 1/10 \\ 13/30 & 11/30 & 19/30 & 1/6 & 13/30 & 4/5 & 0 & 9/10 \\ 7/15 & 8/15 & 4/15 & 11/15 & 1/2 & 1/10 & 9/10 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/15 & 1/10 & 1/6 & 1/5 & 3/10 & 11/30 & 7/15 \\ 1/15 & 0 & 1/6 & 1/10 & 1/5 & 11/30 & 3/10 & 8/15 \\ 1/10 & 1/6 & 0 & 4/15 & 7/30 & 1/5 & 7/15 & 11/30 \\ 1/6 & 1/10 & 4/15 & 0 & 7/30 & 7/15 & 1/5 & 19/30 \\ 1/5 & 1/5 & 7/30 & 7/30 & 0 & 11/30 & 11/30 & 1/2 \\ 3/10 & 11/30 & 1/5 & 7/15 & 11/30 & 0 & 2/3 & 1/6 \\ 11/30 & 3/10 & 7/15 & 1/5 & 11/30 & 2/3 & 0 & 5/6 \\ 7/15 & 8/15 & 11/30 & 19/30 & 1/2 & 1/6 & 5/6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/10 & 2/15 & 3/10 & 3/10 & 1/3 & 2/5 & 7/15 \\ 1/10 & 0 & 7/30 & 1/5 & 2/5 & 3/10 & 3/10 & 17/30 \\ 2/15 & 7/30 & 0 & 13/30 & 1/6 & 2/5 & 8/15 & 1/3 \\ 3/10 & 1/5 & 13/30 & 0 & 3/5 & 3/10 & 1/10 & 23/30 \\ 3/10 & 2/5 & 1/6 & 3/5 & 0 & 1/2 & 7/10 & 1/6 \\ 1/3 & 3/10 & 2/5 & 3/10 & 1/2 & 0 & 1/3 & 3/5 \\ 2/5 & 3/10 & 8/15 & 1/10 & 7/10 & 1/3 & 0 & 13/15 \\ 7/15 & 17/30 & 1/3 & 23/30 & 1/6 & 3/5 & 13/15 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3/4+x & 4x & 1/4+3x & 1/2+2x & 3/4+x \\ 3/4+x & 0 & 3/4+x & 2/3 & 1/2 & 1/3 \\ 4x & 3/4+x & 0 & 1/4-x & 1/2-2x & 3/4-3x \\ 1/4+3x & 2/3 & 1/4-x & 0 & 1/4-x & 1/2-2x \\ 1/2+2x & 1/2 & 1/2-2x & 1/4-x & 0 & 1/4-x \\ 3/4+x & 1/3 & 3/4-3x & 1/2-2x & 1/4-x & 0 \end{pmatrix} \text{ for } 0 \le x \le 1/12$$

$$\begin{pmatrix} 0 & 2/3+x & 2x & 1/2+2x & 1/3+x & 1/2+x \\ 2/3+x & 0 & 2/3-x & 1/2-x & 1/3 & 1/6 \\ 2x & 2/3-x & 0 & 1/2+2x & 1/3-x & 1/2-x \\ 1/2+2x & 1/2-x & 1/2+2x & 0 & 1/2+x & 1/2 \\ 1/3+x & 1/3 & 1/3-x & 1/2+x & 0 & 1/6 \\ 1/2+x & 1/6 & 1/2-x & 1/2 & 1/6 & 0 \end{pmatrix} \text{ for } 0 \le x \le 1/6$$

$$\begin{pmatrix} 0 & 1/2+x & 2x & 2x & 1/2+x & 1/2 \\ 1/2+x & 0 & 1/2 & 1/2-x & 1/3 & 1/3 \\ 2x & 1/2 & 0 & 2x & 1/2-x & 1/2+x \\ 2x & 1/2-x & 2x & 0 & 1/2 & 1/2-x \\ 1/2+x & 1/3 & 1/2-x & 1/2 & 0 & 2/3 \\ 1/2 & 1/3 & 1/2+x & 1/2-x & 2/3 & 0 \end{pmatrix} \text{ for } 0 \le x \le 1/3$$

$$\begin{pmatrix} 0 & 1/2+x & 4x & 1/2+3x & 2x & 1/2+x \\ 1/2+x & 0 & 1/2-x & 1/3 & 1/2 & 1/3 \\ 4x & 1/2-x & 0 & 1/2-x & 2x & 1/2-3x \\ 1/2+3x & 1/3 & 1/2-x & 0 & 1/2+x & 2x \\ 2x & 1/2 & 2x & 1/2+x & 0 & 1/2-x \\ 1/2+x & 1/3 & 1/2-3x & 2x & 1/2-x & 0 \end{pmatrix} \text{ for } 0 \le x \le 1/6$$

$$\begin{pmatrix} 0 & 2/3+x & 2/3+4x & 3x & 1/3+2x & 2/3+x \\ 2/3+x & 0 & 3x & 2/3-2x & 1/3-x & 1/3 \\ 2/3+4x & 3x & 0 & 2/3+x & 1/3+2x & 1/3-x \\ 3x & 2/3-2x & 2/3+x & 0 & 1/3-x & 2/3 \\ 1/3+2x & 1/3-x & 1/3+2x & 1/3-x & 0 & 1/2 \\ 2/3+x & 1/3 & 1/3-x & 2/3 & 1/2 & 0 \end{pmatrix} \text{ for } 0 \le x \le 1/12$$

$$\begin{pmatrix} 0 & 1/21 & 5/42 & 1/6 & 2/7 & 10/21 \\ 1/21 & 0 & 1/6 & 5/42 & 2/7 & 11/21 \\ 5/42 & 1/6 & 0 & 2/7 & 13/42 & 5/14 \\ 1/6 & 5/42 & 2/7 & 0 & 13/42 & 9/14 \\ 2/7 & 2/7 & 13/42 & 13/42 & 0 & 1/2 \\ 10/21 & 11/21 & 5/14 & 9/14 & 1/2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/14 & 1/14 & 8/21 & 8/21 & 1/2 \\ 1/14 & 0 & 1/7 & 13/42 & 19/42 & 19/42 \\ 1/14 & 1/7 & 0 & 19/42 & 13/42 & 23/42 \\ 8/21 & 13/42 & 19/42 & 0 & 16/21 & 2/7 \\ 8/21 & 19/42 & 13/42 & 16/21 & 0 & 5/7 \\ 1/2 & 19/42 & 23/42 & 2/7 & 5/7 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2/15 & 2/15 & 3/10 & 3/10 & 1/2 \\ 2/15 & 0 & 4/15 & 1/6 & 13/30 & 7/15 \\ 2/15 & 4/15 & 0 & 13/30 & 1/6 & 8/15 \\ 3/10 & 1/6 & 13/30 & 0 & 3/5 & 13/30 \\ 3/10 & 13/30 & 1/6 & 3/5 & 0 & 17/30 \\ 1/2 & 7/15 & 8/15 & 13/30 & 17/30 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/21 & 1/7 & 11/42 & 13/42 & 10/21 \\ 1/21 & 0 & 1/7 & 13/42 & 11/42 & 11/21 \\ 1/7 & 1/7 & 0 & 13/42 & 13/42 & 1/2 \\ 11/42 & 13/42 & 13/42 & 0 & 4/7 & 3/14 \\ 13/42 & 11/42 & 13/42 & 4/7 & 0 & 11/14 \\ 10/21 & 11/21 & 1/2 & 3/14 & 11/14 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/10 & 1/10 & 11/30 & 11/30 & 1/2 \\ 1/10 & 0 & 1/5 & 4/15 & 7/15 & 7/15 \\ 1/10 & 1/5 & 0 & 7/15 & 4/15 & 8/15 \\ 11/30 & 4/15 & 7/15 & 0 & 11/15 & 2/5 \\ 11/30 & 7/15 & 4/15 & 11/15 & 0 & 3/5 \\ 1/2 & 7/15 & 8/15 & 2/5 & 3/5 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/14 & 2/21 & 2/7 & 5/14 & 8/21 \\ 1/14 & 0 & 1/6 & 11/42 & 2/7 & 19/42 \\ 2/21 & 1/6 & 0 & 1/3 & 19/42 & 2/7 \\ 2/7 & 11/42 & 1/3 & 0 & 13/42 & 11/21 \\ 5/14 & 2/7 & 19/42 & 13/42 & 0 & 31/42 \\ 8/21 & 19/42 & 2/7 & 11/21 & 31/42 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/15 & 7/60 & 11/60 & 7/20 & 7/15 \\ 1/15 & 0 & 11/60 & 7/60 & 7/20 & 8/15 \\ 7/60 & 11/60 & 0 & 3/10 & 11/30 & 7/20 \\ 11/60 & 7/60 & 3/10 & 0 & 11/30 & 13/20 \\ 7/20 & 7/20 & 11/30 & 11/30 & 0 & 1/2 \\ 7/15 & 8/15 & 7/20 & 13/20 & 1/2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2/15 & 2/15 & 13/30 & 13/30 & 1/2 \\ 2/15 & 0 & 4/15 & 3/10 & 17/30 & 2/5 \\ 2/15 & 4/15 & 0 & 17/30 & 3/10 & 3/5 \\ 13/30 & 3/10 & 17/30 & 0 & 13/15 & 7/30 \\ 13/30 & 17/30 & 3/10 & 13/15 & 0 & 23/30 \\ 1/2 & 2/5 & 3/5 & 7/30 & 23/30 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/6 & 3/14 & 17/42 & 3/7 & 19/42 \\ 1/6 & 0 & 8/21 & 5/21 & 25/42 & 3/7 \\ 3/14 & 8/21 & 0 & 13/21 & 3/14 & 1/2 \\ 17/42 & 5/21 & 13/21 & 0 & 5/6 & 3/7 \\ 3/7 & 25/42 & 3/14 & 5/6 & 0 & 23/42 \\ 19/42 & 3/7 & 1/2 & 3/7 & 23/42 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2/15 & 1/6 & 4/15 & 1/3 & 13/30 \\ 2/15 & 0 & 1/6 & 1/3 & 4/15 & 17/30 \\ 1/6 & 1/6 & 0 & 13/30 & 13/30 & 1/2 \\ 4/15 & 1/3 & 13/30 & 0 & 4/15 & 11/30 \\ 1/3 & 4/15 & 13/30 & 4/15 & 0 & 19/30 \\ 13/30 & 17/30 & 1/2 & 11/30 & 19/30 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 7/60 & 1/6 & 17/60 & 2/5 & 5/12 \\ 7/60 & 0 & 17/60 & 2/5 & 5/12 & 3/10 \\ 1/6 & 17/60 & 0 & 7/60 & 2/5 & 7/12 \\ 17/60 & 2/5 & 7/60 & 0 & 5/12 & 7/10 \\ 2/5 & 5/12 & 2/5 & 5/12 & 0 & 1/2 \\ 5/12 & 3/10 & 7/12 & 7/10 & 1/2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/15 & 1/15 & 9/20 & 9/20 & 1/2 \\ 1/15 & 0 & 2/15 & 23/60 & 31/60 & 9/20 \\ 1/15 & 2/15 & 0 & 31/60 & 23/60 & 11/20 \\ 9/20 & 23/60 & 31/60 & 0 & 9/10 & 7/30 \\ 9/20 & 31/60 & 23/60 & 9/10 & 0 & 23/30 \\ 1/2 & 9/20 & 11/20 & 7/30 & 23/30 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/6 & 1/6 & 11/30 & 11/30 & 1/2 \\ 1/6 & 0 & 4/15 & 1/3 & 8/15 & 11/30 \\ 1/6 & 4/15 & 0 & 8/15 & 1/3 & 19/30 \\ 11/30 & 1/3 & 8/15 & 0 & 8/15 & 7/30 \\ 11/30 & 8/15 & 1/3 & 8/15 & 0 & 23/30 \\ 1/2 & 11/30 & 19/30 & 7/30 & 23/30 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2/15 & 2/15 & 1/5 & 4/15 & 13/30 \\ 2/15 & 0 & 4/15 & 1/5 & 2/15 & 17/30 \\ 2/15 & 4/15 & 0 & 4/15 & 2/5 & 3/10 \\ 1/5 & 1/5 & 4/15 & 0 & 4/15 & 1/2 \\ 4/15 & 2/15 & 2/5 & 4/15 & 0 & 7/10 \\ 13/30 & 17/30 & 3/10 & 1/2 & 7/10 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2/21 & 3/14 & 5/14 & 3/7 & 10/21 \\ 2/21 & 0 & 13/42 & 11/42 & 8/21 & 4/7 \\ 3/14 & 13/42 & 0 & 4/7 & 23/42 & 11/42 \\ 5/14 & 11/42 & 4/7 & 0 & 13/42 & 5/6 \\ 3/7 & 8/21 & 23/42 & 13/42 & 0 & 2/3 \\ 10/21 & 4/7 & 11/42 & 5/6 & 2/3 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/15 & 3/20 & 19/60 & 23/60 & 7/15 \\ 1/15 & 0 & 3/20 & 23/60 & 19/60 & 8/15 \\ 3/20 & 3/20 & 0 & 11/30 & 11/30 & 1/2 \\ 19/60 & 23/60 & 11/30 & 0 & 7/10 & 3/20 \\ 23/60 & 19/60 & 11/30 & 7/10 & 0 & 17/20 \\ 7/15 & 8/15 & 1/2 & 3/20 & 17/20 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/10 & 11/60 & 17/60 & 13/30 & 9/20 \\ 1/10 & 0 & 17/60 & 11/60 & 13/30 & 11/20 \\ 11/60 & 17/60 & 0 & 7/15 & 9/20 & 4/15 \\ 17/60 & 11/60 & 7/15 & 0 & 9/20 & 11/15 \\ 13/30 & 13/30 & 9/20 & 9/20 & 0 & 1/2 \\ 9/20 & 11/20 & 4/15 & 11/15 & 1/2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2/21 & 2/21 & 5/14 & 5/14 & 1/2 \\ 2/21 & 0 & 4/21 & 11/42 & 19/42 & 3/7 \\ 2/21 & 4/21 & 0 & 19/42 & 11/42 & 4/7 \\ 5/14 & 11/42 & 19/42 & 0 & 5/7 & 11/42 \\ 5/14 & 19/42 & 11/42 & 5/7 & 0 & 31/42 \\ 1/2 & 3/7 & 4/7 & 11/42 & 31/42 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/10 & 1/10 & 5/12 & 5/12 & 1/2 \\ 1/10 & 0 & 1/5 & 19/60 & 31/60 & 5/12 \\ 1/10 & 1/5 & 0 & 31/60 & 19/60 & 7/12 \\ 5/12 & 19/60 & 31/60 & 0 & 5/6 & 1/5 \\ 5/12 & 31/60 & 19/60 & 5/6 & 0 & 4/5 \\ 1/2 & 5/12 & 7/12 & 1/5 & 4/5 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/15 & 7/30 & 3/10 & 2/5 & 7/15 \\ 1/15 & 0 & 3/10 & 7/30 & 2/5 & 8/15 \\ 7/30 & 3/10 & 0 & 8/15 & 13/30 & 7/30 \\ 3/10 & 7/30 & 8/15 & 0 & 13/30 & 23/30 \\ 2/5 & 2/5 & 13/30 & 13/30 & 0 & 1/2 \\ 7/15 & 8/15 & 7/30 & 23/30 & 1/2 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 5/42 & 5/21 & 17/42 & 3/7 & 10/21 \\ 5/42 & 0 & 5/14 & 2/7 & 17/42 & 25/42 \\ 5/21 & 5/14 & 0 & 9/14 & 1/2 & 5/21 \\ 17/42 & 2/7 & 9/14 & 0 & 17/42 & 37/42 \\ 3/7 & 17/42 & 1/2 & 17/42 & 0 & 4/7 \\ 10/21 & 25/42 & 5/21 & 37/42 & 4/7 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/15 & 2/15 & 1/6 & 7/30 & 7/15 \\ 1/15 & 0 & 2/15 & 7/30 & 1/6 & 8/15 \\ 2/15 & 2/15 & 0 & 7/30 & 7/30 & 1/2 \\ 1/6 & 7/30 & 7/30 & 0 & 2/5 & 3/10 \\ 7/30 & 1/6 & 7/30 & 2/5 & 0 & 7/10 \\ 7/15 & 8/15 & 1/2 & 3/10 & 7/10 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2/3+x & 1/6+2x & 1/3+2x & 1/2+2x \\ 2/3+x & 0 & 5/6-x & 1/3-x & 1/6+x \\ 1/6+2x & 5/6-x & 0 & 1/2+2x & 2/3-2x \\ 1/3+2x & 1/3-x & 1/2+2x & 0 & 1/6+2x \\ 1/2+2x & 1/6+x & 2/3-2x & 1/6+2x & 0 \end{pmatrix} \text{ for } 0 \le x \le 1/6$$

$$\begin{pmatrix} 0 & 2/3+x & 1/6+3x & 1/6+2x & 1/2+x \\ 2/3+x & 0 & 5/6-2x & 1/2-x & 1/6 \\ 1/6+3x & 5/6-2x & 0 & 1/3+3x & 2/3 \\ 1/6+2x & 1/2-x & 1/3+3x & 0 & 1/3-x \\ 1/2+x & 1/6 & 2/3 & 1/3-x & 0 \end{pmatrix} \text{ for } 0 \le x \le 1/6$$

$$\begin{pmatrix} 0 & 7/60 & 13/60 & 7/30 & 7/20 \\ 7/60 & 0 & 7/30 & 7/20 & 7/15 \\ 13/60 & 7/30 & 0 & 19/60 & 2/5 \\ 7/30 & 7/20 & 19/60 & 0 & 7/60 \\ 7/20 & 7/15 & 2/5 & 7/60 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 4/21 & 2/7 & 17/42 & 3/7 \\ 4/21 & 0 & 3/7 & 25/42 & 2/7 \\ 2/7 & 3/7 & 0 & 11/42 & 10/21 \\ 17/42 & 25/42 & 11/42 & 0 & 31/42 \\ 3/7 & 2/7 & 10/21 & 31/42 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/10 & 1/3 & 7/20 & 9/20 \\ 1/10 & 0 & 2/5 & 9/20 & 7/20 \\ 1/3 & 2/5 & 0 & 1/4 & 13/20 \\ 7/20 & 9/20 & 1/4 & 0 & 4/5 \\ 9/20 & 7/20 & 13/20 & 4/5 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/7 & 4/21 & 5/21 & 19/42 \\ 1/7 & 0 & 5/21 & 8/21 & 25/42 \\ 4/21 & 5/21 & 0 & 2/7 & 19/42 \\ 5/21 & 8/21 & 2/7 & 0 & 3/14 \\ 19/42 & 25/42 & 19/42 & 3/14 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/5 & 2/5 & 2/5 & 7/15 \\ 1/5 & 0 & 8/15 & 3/5 & 3/5 \\ 2/5 & 8/15 & 0 & 4/15 & 8/15 \\ 2/5 & 3/5 & 4/15 & 0 & 4/15 \\ 7/15 & 3/5 & 8/15 & 4/15 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/7 & 1/3 & 3/7 & 3/7 \\ 1/7 & 0 & 3/7 & 1/3 & 4/7 \\ 1/3 & 3/7 & 0 & 3/7 & 2/7 \\ 3/7 & 1/3 & 3/7 & 0 & 5/7 \\ 3/7 & 4/7 & 2/7 & 5/7 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/14 & 5/21 & 3/7 & 10/21 \\ 1/14 & 0 & 13/42 & 17/42 & 17/42 \\ 5/21 & 13/42 & 0 & 11/21 & 5/7 \\ 3/7 & 17/42 & 11/21 & 0 & 8/21 \\ 10/21 & 17/42 & 5/7 & 8/21 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/6 & 11/60 & 11/60 & 7/20 \\ 1/6 & 0 & 17/60 & 7/20 & 11/60 \\ 11/60 & 17/60 & 0 & 1/5 & 13/30 \\ 11/60 & 7/20 & 1/5 & 0 & 8/15 \\ 7/20 & 11/60 & 13/30 & 8/15 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2/15 & 2/15 & 1/5 & 1/5 \\ 2/15 & 0 & 4/15 & 1/5 & 4/15 \\ 2/15 & 4/15 & 0 & 4/15 & 1/5 \\ 1/5 & 1/5 & 4/15 & 0 & 2/5 \\ 1/5 & 4/15 & 1/5 & 2/5 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2/15 & 3/10 & 1/3 & 2/5 \\ 2/15 & 0 & 13/30 & 2/5 & 1/2 \\ 3/10 & 13/30 & 0 & 3/10 & 7/30 \\ 1/3 & 2/5 & 3/10 & 0 & 8/15 \\ 2/5 & 1/2 & 7/30 & 8/15 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 11/60 & 1/5 & 4/15 & 9/20 \\ 11/60 & 0 & 13/60 & 9/20 & 19/30 \\ 1/5 & 13/60 & 0 & 11/30 & 31/60 \\ 4/15 & 9/20 & 11/30 & 0 & 11/60 \\ 9/20 & 19/30 & 31/60 & 11/60 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/10 & 19/60 & 11/30 & 5/12 \\ 1/10 & 0 & 5/12 & 13/30 & 19/60 \\ 19/60 & 5/12 & 0 & 1/4 & 11/15 \\ 11/30 & 13/30 & 1/4 & 0 & 13/20 \\ 5/12 & 19/60 & 11/15 & 13/20 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3/20 & 3/20 & 7/20 & 7/20 \\ 3/20 & 0 & 3/10 & 1/3 & 2/5 \\ 3/20 & 3/10 & 0 & 2/5 & 1/3 \\ 7/20 & 1/3 & 2/5 & 0 & 7/10 \\ 7/20 & 2/5 & 1/3 & 7/10 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 3/20 & 11/60 & 7/30 & 23/60 \\ 3/20 & 0 & 7/30 & 23/60 & 8/15 \\ 11/60 & 7/30 & 0 & 17/60 & 2/5 \\ 7/30 & 23/60 & 17/60 & 0 & 3/20 \\ 23/60 & 8/15 & 2/5 & 3/20 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/10 & 1/6 & 4/15 & 11/30 \\ 1/10 & 0 & 4/15 & 7/30 & 4/15 \\ 1/6 & 4/15 & 0 & 11/30 & 8/15 \\ 4/15 & 7/30 & 11/30 & 0 & 3/10 \\ 11/30 & 4/15 & 8/15 & 3/10 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2/15 & 1/6 & 1/6 & 7/15 \\ 2/15 & 0 & 7/30 & 3/10 & 3/5 \\ 1/6 & 7/30 & 0 & 1/5 & 13/30 \\ 1/6 & 3/10 & 1/5 & 0 & 3/10 \\ 7/15 & 3/5 & 13/30 & 3/10 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2/15 & 3/20 & 17/60 & 2/5 \\ 2/15 & 0 & 17/60 & 3/20 & 13/30 \\ 3/20 & 17/60 & 0 & 13/30 & 23/60 \\ 17/60 & 3/20 & 13/30 & 0 & 29/60 \\ 2/5 & 13/30 & 23/60 & 29/60 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/15 & 11/30 & 23/60 & 9/20 \\ 1/15 & 0 & 2/5 & 9/20 & 23/60 \\ 11/30 & 2/5 & 0 & 19/60 & 37/60 \\ 23/60 & 9/20 & 19/60 & 0 & 5/6 \\ 9/20 & 23/60 & 37/60 & 5/6 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/18 & 1/9 & 1/6 & 5/18 \\ 1/18 & 0 & 5/18 & 1/9 & 2/9 \\ 1/9 & 5/18 & 0 & 1/2 & 13/18 \\ 1/6 & 1/9 & 1/2 & 0 & 1/9 \\ 5/18 & 2/9 & 13/18 & 1/9 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 7/60 & 7/30 & 3/10 & 5/12 \\ 7/60 & 0 & 1/4 & 5/12 & 8/15 \\ 7/30 & 1/4 & 0 & 11/30 & 9/20 \\ 3/10 & 5/12 & 11/30 & 0 & 7/60 \\ 5/12 & 8/15 & 9/20 & 7/60 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2/15 & 1/5 & 7/20 & 29/60 \\ 2/15 & 0 & 7/30 & 29/60 & 7/20 \\ 1/5 & 7/30 & 0 & 23/60 & 29/60 \\ 7/20 & 29/60 & 23/60 & 0 & 5/6 \\ 29/60 & 7/20 & 29/60 & 5/6 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2/15 & 7/30 & 3/10 & 1/2 \\ 2/15 & 0 & 7/30 & 13/30 & 7/15 \\ 7/30 & 7/30 & 0 & 2/5 & 7/10 \\ 3/10 & 13/30 & 2/5 & 0 & 17/30 \\ 1/2 & 7/15 & 7/10 & 17/30 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3/20 & 1/5 & 3/10 & 9/20 \\ 3/20 & 0 & 1/4 & 9/20 & 3/5 \\ 1/5 & 1/4 & 0 & 1/3 & 9/20 \\ 3/10 & 9/20 & 1/3 & 0 & 3/20 \\ 9/20 & 3/5 & 9/20 & 3/20 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/7 & 1/7 & 2/7 & 2/7 \\ 1/7 & 0 & 2/7 & 2/7 & 1/3 \\ 1/7 & 2/7 & 0 & 1/3 & 2/7 \\ 2/7 & 2/7 & 1/3 & 0 & 4/7 \\ 2/7 & 1/3 & 2/7 & 4/7 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3/20 & 1/6 & 13/60 & 19/60 \\ 3/20 & 0 & 19/60 & 1/5 & 7/15 \\ 1/6 & 19/60 & 0 & 19/60 & 3/20 \\ 13/60 & 1/5 & 19/60 & 0 & 13/30 \\ 19/60 & 7/15 & 3/20 & 13/30 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2/15 & 2/5 & 13/30 & 1/2 \\ 2/15 & 0 & 1/2 & 17/30 & 2/5 \\ 2/5 & 1/2 & 0 & 7/30 & 8/15 \\ 13/30 & 17/30 & 7/30 & 0 & 23/30 \\ 1/2 & 2/5 & 8/15 & 23/30 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/15 & 7/20 & 2/5 & 5/12 \\ 1/15 & 0 & 5/12 & 13/30 & 7/20 \\ 7/20 & 5/12 & 0 & 19/60 & 23/30 \\ 2/5 & 13/30 & 19/60 & 0 & 37/60 \\ 5/12 & 7/20 & 23/30 & 37/60 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2/15 & 1/6 & 1/6 & 7/15 \\ 2/15 & 0 & 7/30 & 3/10 & 1/2 \\ 1/6 & 7/30 & 0 & 1/5 & 3/10 \\ 1/6 & 3/10 & 1/5 & 0 & 13/30 \\ 7/15 & 1/2 & 3/10 & 13/30 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3/20 & 13/60 & 5/12 & 13/30 \\ 3/20 & 0 & 7/30 & 4/15 & 7/12 \\ 13/60 & 7/30 & 0 & 2/5 & 29/60 \\ 5/12 & 4/15 & 2/5 & 0 & 17/20 \\ 13/30 & 7/12 & 29/60 & 17/20 & 0 \end{pmatrix}$$

11.3. **Regge orbits of rational-angle 4-line configurations.** Here we list representatives of the $\mathfrak{R}^{\pm}$-orbits of $\mathbb{R}^3$-realizable 4×4 rational-angle matrices, with each angle measured in units of π. We exclude any orbit containing a representative obtained from a matrix in Section 11.2 or from a perpendicular configuration. In each orbit, we choose the representative for which the sum of the denominators of the matrix entries is smallest, and among those, we choose the one that is lexicographically smallest.

$$\begin{pmatrix} 0 & 1/20 & 11/60 & 11/30 \\ 1/20 & 0 & 1/5 & 5/12 \\ 11/60 & 1/5 & 0 & 7/20 \\ 11/30 & 5/12 & 7/20 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/14 & 13/84 & 25/84 \\ 1/14 & 0 & 19/84 & 9/28 \\ 13/84 & 19/84 & 0 & 2/7 \\ 25/84 & 9/28 & 2/7 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 11/90 & 4/15 & 22/45 \\ 11/90 & 0 & 7/18 & 17/30 \\ 4/15 & 7/18 & 0 & 1/3 \\ 22/45 & 17/30 & 1/3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3/20 & 19/120 & 5/24 \\ 3/20 & 0 & 37/120 & 13/40 \\ 19/120 & 37/120 & 0 & 3/20 \\ 5/24 & 13/40 & 3/20 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1/15 & 5/18 & 1/3 \\ 1/15 & 0 & 31/90 & 17/45 \\ 5/18 & 31/90 & 0 & 7/30 \\ 1/3 & 17/45 & 7/30 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2/15 & 3/14 & 44/105 \\ 2/15 & 0 & 73/210 & 3/7 \\ 3/14 & 73/210 & 0 & 13/30 \\ 44/105 & 3/7 & 13/30 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2/35 & 2/7 & 3/10 \\ 2/35 & 0 & 1/3 & 5/14 \\ 2/7 & 1/3 & 0 & 11/70 \\ 3/10 & 5/14 & 11/70 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 7/120 & 3/10 & 43/120 \\ 7/120 & 0 & 43/120 & 2/5 \\ 3/10 & 43/120 & 0 & 9/40 \\ 43/120 & 2/5 & 9/40 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1/18 & 1/3 & 7/15 \\ 1/18 & 0 & 11/30 & 37/90 \\ 1/3 & 11/30 & 0 & 29/45 \\ 7/15 & 37/90 & 29/45 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1/10 & 3/14 & 27/70 \\ 1/10 & 0 & 11/35 & 3/7 \\ 3/14 & 11/35 & 0 & 1/3 \\ 27/70 & 3/7 & 1/3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 5/84 & 1/4 & 5/14 \\ 5/84 & 0 & 2/7 & 5/12 \\ 1/4 & 2/7 & 0 & 23/84 \\ 5/14 & 5/12 & 23/84 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3/14 & 3/10 & 29/70 \\ 3/14 & 0 & 18/35 & 1/3 \\ 3/10 & 18/35 & 0 & 4/7 \\ 29/70 & 1/3 & 4/7 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 8/45 & 3/10 & 1/3 \\ 8/45 & 0 & 43/90 & 2/5 \\ 3/10 & 43/90 & 0 & 31/90 \\ 1/3 & 2/5 & 31/90 & 0 \end{pmatrix}$$

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