GALOIS POINTS ON VARIETIES

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Abstract. A field K is ample if for every geometrically integral K-variety V with a smooth K-point, $V(K)$ is Zariski dense in V. A field K is Galois-potent if every geometrically integral K-variety has a closed point whose residue field is Galois over K. We prove that every ample field is Galois-potent. But we construct also non-ample Galois-potent fields; in fact, every field has a regular extension with these properties.

1. Introduction

Definition 1.1. Let X be a variety over a field K. By a Galois point on X, we mean a closed point whose residue field is Galois over K. We say that K is Galois-potent if every geometrically integral K-variety has a Galois point.

Question 1.2. Is every field Galois-potent?

We do not even know if $\mathbb{Q}$ is Galois-potent. On the other hand, we have the following definition of Florian Pop:

Definition 1.3 (cf. [Pop96, p. 2] and [Jar11, Chapter 5])**.** A field K is called ample (or large or anti-Mordellic) if for every geometrically integral K-variety V with a smooth K-point, $V(K)$ is Zariski dense in V.

Pseudo-algebraically closed (PAC) fields and Henselian fields (e.g., $\mathbb{Q}_p$) are ample; see [Jar11, Chapter 5] for these and many more examples. Our first main theorem is the following:

Theorem 1.4. *Ample fields are Galois-potent.*

Some non-ample fields too are Galois-potent. For example, finite fields are non-ample, but have abelian absolute Galois group and hence trivially are Galois-potent. Less trivially, we can also construct *infinite* non-ample fields that are Galois-potent: in Section 5 we will prove the following.

Theorem 1.5. *Every field admits a regular extension that is Galois-potent but not ample.*

Bary-Soroker and Fehm in [BSF13, Section 2.2] write that "all infinite non-ample fields appearing in the literature are Hilbertian". The non-ample fields we construct in the proof of Theorem 1.5 turn out to be Hilbertian too (Remark 5.4), and in particular they have non-abelian absolute Galois group.

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Question 1.6. Are there also infinite non-ample fields with abelian absolute Galois group?

If the answer to Question 1.6 were yes, it would immediately yield another example of a non-ample Galois-potent field. But there are several questions and conjectures in the literature that each imply a negative answer to Question 1.6, as we now explain. Let G_K be the absolute Galois group of K.

- Is every infinite non-ample field Hilbertian? (Cf. [BSF13, Section 2.2] again.) A Hilbertian field K cannot have an abelian G_K.
- Is every infinite field K with topologically finitely generated G_K ample? (See the paragraph before Theorem 5.5 of [JK10], or see [BSF13, 4.2, Question III].) By [Koe01], if G_K is abelian, either K is henselian and hence ample, or G_K is procyclic and in particular topologically finitely generated, and hence ample if the question at the start of this bulleted item has a positive answer.
- Is every infinite field with prosolvable G_K ample? (See [BSF13, 4.1, Question II].) If so, then every infinite field with abelian G_K is ample too.

2. Notation

Let K be a field. Let $\overline{K}$ be an algebraic closure of K. By a K-variety we mean a separated scheme of finite type over K. Given a K-variety X and a finite extension $L \supset K$, let X_L be the L-variety $X \times_K L$; similarly, if ϕ is a K-morphism, let ϕ_L be its base change. If X is an integral K-variety, let $K(X)$ denote its function field. By a K-curve, we mean a smooth geometrically integral K-variety of dimension 1. By the absolute genus g_X of a K-curve X, we mean the genus of the smooth projective model of $X_{\overline{K}}$. Given a curve X and $n \geq 1$, let $X^{(n)}$ denote the n^{th} symmetric power, defined as the (variety) quotient of X^n by the symmetric group S_n.

3. Restriction of scalars

Let $K \subseteq L$ be a finite extension of fields. For any quasi-projective L-variety X, the functor sending each K-scheme S to the set $X(S_L)$ is representable by a K-variety $\operatorname{Res}_{L/K} X$ called the restriction of scalars (cf. [BLR90, §7.6, Theorem 4]). If, in addition, L is separable over K, then $\operatorname{Res}_{L/K} X$ after base field extension becomes isomorphic to a product of $[L : K]$ conjugates of X; in particular, if X is geometrically integral and smooth of dimension d over L, then $\operatorname{Res}_{L/K} X$ is geometrically integral and smooth of dimension $[L : K]d$ over K.

4. Ample fields are Galois-potent

In this section we prove Theorem 1.4.

Proposition 4.1. *Let K be an ample field. Let X be a geometrically integral K-variety. Let N be a finite separable extension of K. If X has a smooth N-point, then X has a point with residue field N.*

Proof. Replacing X by a Zariski-open subvariety, we may assume that X is smooth and affine. For each finite separable extension L of K, define $X^L := \operatorname{Res}_{L/K}(X_L)$. Thus $X^L(S) = X_L(S_L) = X(S_L)$ for any K-scheme S, and in particular, $X^L(K) = X(L)$. If $L \subseteq L'$ are two such extensions, the natural map $X(S_L) \to X(S_{L'})$ can be rewritten as $X^L(S) \to X^{L'}(S)$, and it is functorial in S, so Yoneda's lemma yields a K-morphism $X^L \to X^{L'}$.

We have $X^N(K) = X(N)$, which is nonempty by assumption. Also, K is ample, and X^N is smooth and geometrically integral, so $X^N(K)$ is Zariski-dense in X^N. There are only finitely many fields L with $K \subseteq L \subsetneq N$, and for each such L, we have $\dim X^L < \dim X^N$. Thus X^N has a K-point outside the image of every morphism $X^L \to X^N$. In other words, $X(N)$ has a element outside $X(L)$ for every L. For this element, the image of $\operatorname{Spec} N \to X$ is a closed point with residue field N. □

Proof of Theorem 1.4. Let K be an ample field, and let X be any geometrically integral K-variety. Choose a finite separable extension N of K such that $X(N) \neq \emptyset$. Enlarge N to assume that N is Galois over K. By Proposition 4.1, X has a closed point with residue field N. □

5. Infinite Galois-potent fields that are not ample

In this section we prove Theorem 1.5.

Lemma 5.1. *Let K be an algebraically closed field. Let X, Y, C be K-curves with $g_C > 1$. Any rational map $X \times Y \dashrightarrow C$ factors through one of the projections to X or Y.*

Proof. A rational map $\phi\colon X \times Y \dashrightarrow C$ may be viewed as an algebraic family of rational maps $X \dashrightarrow C$ parametrized by (an open subvariety of) Y. But the de Franchis–Severi theorem [Sam66, Théorème 1] implies that there are no nonconstant algebraic families of nonconstant rational maps $X \dashrightarrow C$. Thus either the rational maps in the family are all the same, in which case ϕ factors through the first projection, or each rational map in the family is constant, in which case ϕ factors through the second projection. □

Lemma 5.2. *Let X be a curve over a field K. Let $F = K(X^{(2)})$. Then X has a point over a quadratic extension of F, and $C(F) = C(K)$ for every K-curve C of absolute genus > 1.*

Proof. Either projection $X \times X \to X$ gives a point of X over the quadratic extension $K(X \times X)$ of $K(X^{(2)}) = F$.

Let $c \in C(F)$. Then c corresponds to a rational map $X^{(2)} \dashrightarrow C$. Composing $X \times X \to X^{(2)}$ with such a rational map yields a rational map $\phi\colon X \times X \dashrightarrow C$. By Lemma 5.1, $\phi_{\overline{K}}$ is constant on all vertical copies of $X_{\overline{K}}$ or constant on all horizontal copies of $X_{\overline{K}}$. Since $\phi_{\overline{K}}$ is S_2-invariant, it is constant on all vertical and horizontal copies of $X_{\overline{K}}$. Thus $\phi_{\overline{K}}$ is constant. Hence ϕ is constant. Equivalently, $c \in C(K)$. □

Lemma 5.3. *Every field K admits a regular extension K' such that*

(i) *Every K-curve X has a point over an at most quadratic extension of K' (possibly depending on X).*
(ii) *For every K-curve C of absolute genus > 1, we have $C(K') = C(K)$.*

Proof. Let $(X_\alpha)_{\alpha<\tau}$ be a well-ordering of the set of K-curves up to isomorphism, indexed by an ordinal τ. For $\alpha \le \tau$, define K_α by transfinite induction as follows:

- Let $K_0 := K$.
- For each $\alpha < \tau$, define $K_{\alpha+1} := K_\alpha(X_\alpha^{(2)})$;
- If α is a limit ordinal, define $K_\alpha := \varinjlim_{\beta<\alpha} K_\beta$.

Let $K' := K_\tau$. Then K' is regular over K by [FJ08, Corollary 2.6.5(d)], whose proof remains valid when the cardinal m is replaced by an ordinal τ.

(i) Any K-curve X is X_α for some $\alpha < \tau$. Lemma 5.2 shows that X has a point over a quadratic extension of $K_{\alpha+1}$, and hence also a point over an at most quadratic extension of K'.
(ii) Let C be a K-curve of absolute genus > 1. By Lemma 5.2 and induction, $C(K_\alpha) = C(K)$ for all $\alpha \le \tau$. In particular, $C(K') = C(K)$. □

Proof of Theorem 1.5. Let K be the given field. Construct a sequence of fields $L_0, L_1, \ldots$ inductively as follows: let $L_0 := K(t)$ with t transcendental over K, and let L_{i+1} be the regular extension of L_i given by Lemma 5.3. Let $L_\infty := \varinjlim L_i$.

Let X be an L_∞-curve. Then X is definable over some L_i. The conditions in Lemma 5.3 imply that X has a point over an at most quadratic extension of L_{i+1}, and hence a point over an at most quadratic extension of L_∞. Every geometrically integral L_∞-variety other than a point contains an L_∞-curve, so L_∞ is Galois-potent.

Choose a non-isotrivial curve C of absolute genus > 1 over $L_0 = K(t)$ such that C has a smooth L_0-point; then $C(L_0)$ is finite [Sam66, Théorème 4]. The conditions in Lemma 5.3 imply that $C(L_0) = C(L_1) = \cdots$, so $C(L_\infty) = C(L_0)$, which is finite. Thus L_∞ is not ample. □

Remark 5.4. The field L_∞ constructed in the proof of Theorem 1.5 is Hilbertian, because of the following two facts:

(a) Any finitely generated transcendental extension of a field is Hilbertian [FJ08, Theorem 13.4.2].
(b) If $(K_\alpha)_{\alpha \le \tau}$ is an ascending tower of fields indexed by an ordinal τ, and K_α is Hilbertian for each $\alpha < \tau$, and $K_{\alpha+1}$ is a regular extension of K_α for each $\alpha < \tau$, and $K_\alpha = \varinjlim_{\beta<\alpha} K_\beta$ for each limit ordinal $\alpha \le \tau$, then K_τ is Hilbertian.

(Fact (b) appears as [FJ08, Chapter 12, Exercise 2], but the hypothesis $K_\alpha = \varinjlim_{\beta<\alpha} K_\beta$ is missing there.)

Proof of (b). Given irreducible polynomials $f_i(\vec{t}, x)$ over K_τ for $i = 1, \ldots, r$, each separable in x, we must find infinitely many $\vec{a}$ over K_τ such that the specializations $f_i(\vec{a}, x)$ for $i = 1, \ldots, r$ are irreducible over K_τ. There exists $\alpha < \tau$ such that all the f_i are defined over K_α. Since K_α is Hilbertian, there exist infinitely many $\vec{a}$ over K_α such that $f_i(\vec{a}, x)$ for each i is irreducible over K_α. These specializations are irreducible over K_τ as well, since K_τ is a regular extension of K_α by [FJ08, Corollary 2.6.5(d)]. □

Remark 5.5. Padmavathi Srinivasan has constructed a field K with stronger properties than the L_∞ in the proof of Theorem 1.5, namely that $C(K)$ is finite for every K-curve C of absolute genus > 1, while there is a degree 2 field extension $L \supset K$ that is ample [Sri15].

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