THE VALUATION OF THE DISCRIMINANT OF A HYPERSURFACE

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Abstract. Let R be a discrete valuation ring, with valuation $v\colon R \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$ and residue field k. Let H be a hypersurface $\operatorname{Proj} R[x_0,\ldots,x_n]/(f)$. Let H_k be the special fiber, and let $(H_k)_{\mathrm{sing}}$ be its singular subscheme. Let $\Delta(f)$ be the discriminant of f. We use Zariski's main theorem and degeneration arguments to prove that $v(\Delta(f)) = 1$ if and only if H is regular and $(H_k)_{\mathrm{sing}}$ consists of a nondegenerate double point over k. We also give lower bounds on $v(\Delta(f))$ when H_k has multiple singularities or a positive-dimensional singularity.

1. Introduction

Throughout the paper, R denotes a discrete valuation ring, with valuation $v\colon R \twoheadrightarrow \mathbb{Z}\cup\{\infty\}$, maximal ideal $\mathfrak{m} = (\pi)$, and residue field k (except in a few places where k is an arbitrary field).

Let $E \subset \mathbb{P}^2_R$ be defined by a Weierstrass equation, with generic fiber an elliptic curve. If the discriminant of the equation has valuation 1, then E is regular and the singular locus of its special fiber consists of a node; this follows from Tate's algorithm [Tat75], for example; see also [Sil94, Lemma IV.9.5(a)]. Our main theorem (Theorem 1.1) generalizes this to hypersurfaces of arbitrary degree and dimension (terminology will be explained later).

Theorem 1.1. *Let $f \in R[x_0,\ldots,x_n]$ be a homogeneous polynomial. Let $\Delta(f)$ be its discriminant. Let $H = \operatorname{Proj} R[x_0,\ldots,x_n]/(f)$. Then the following are equivalent:*

(i) $v(\Delta(f)) = 1$;
(ii) *H is regular, and $(H_k)_{\mathrm{sing}}$ consists of a nondegenerate double point in $H(k)$.*

We also prove that if $(H_k)_{\mathrm{sing}}$ consists of r isolated closed points, then $v(\Delta(f)) \ge r$ (Theorem 6.2). If $\dim (H_k)_{\mathrm{sing}} \ge 1$, we show that H_k is a limit of hypersurfaces whose singular subscheme is finite but with many points, and we combine this and an argument using the Greenberg functor to deduce that $v(\Delta(f)) \ge \max(\lfloor (\deg f - 1)/2 \rfloor, 2)$ (Theorem 8.4).

2. Discriminant

Fix $n \ge 1$ and $d \ge 2$. Let $x^{\mathbf{i}}$ range over the degree d monomials in $\mathbb{Z}[x_0,\ldots,x_n]$, and let $a_{\mathbf{i}}$ be independent indeterminates, so that $F := \sum_{\mathbf{i}} a_{\mathbf{i}} x^{\mathbf{i}}$ is the generic degree d homogeneous polynomial in $x_0,\ldots,x_n$. Then the affine space $\mathbb{A}^N := \operatorname{Spec} \mathbb{Z}[\{a_{\mathbf{i}}\}]$ may be viewed as a moduli space for hypersurfaces (one could also remove the origin, or projectivize as in [Sai12, §2.4]). Let $\mathcal{H} \subset \mathbb{P}^n \times \mathbb{A}^N$ be the closed subscheme defined by $F = 0$, so the projection $\phi\colon \mathcal{H} \to \mathbb{A}^N$ is the universal hypersurface. Let $\mathcal{H}_{\mathrm{sing}}$ be the relative singular subscheme, the closed subscheme

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defined by $F = \partial F/\partial x_0 = \cdots = \partial F/\partial x_n = 0$. More precisely, $\mathcal{H}_{\text{sing}}$ is the locus of points where ϕ is not smooth of relative dimension $n-1$.

The other projection $\mathcal{H}_{\text{sing}} \to \mathbb{P}^n$ is a rank $N-n-1$ vector bundle since the equations $F = \partial F/\partial x_0 = \cdots = \partial F/\partial x_n = 0$ are linear in the $a_{\mathbf{i}}$ and independent above each point of $\mathbb{P}^n$ except for the Euler relation $d \cdot F = \sum x_i(\partial F/\partial x_i)$. Thus $\mathcal{H}_{\text{sing}}$ is integral and smooth of relative dimension $N-1$ over $\mathbb{Z}$. Its scheme-theoretic image under the proper morphism ϕ is a closed subscheme $D \subset \mathbb{A}^N$, the locus parametrizing singular hypersurfaces. In fact, $D \subset \mathbb{A}^N$ is a divisor and the restriction $\mathcal{H}_{\text{sing}} \to D$ of ϕ is birational (cf. [Sai12, §2.9]); this is a Bertini-type statement saying essentially that among hypersurfaces singular at a point, most have singular subscheme consisting of just that point. Thus $D \subset \mathbb{A}^N$ is the zero locus of some polynomial $\Delta \in \mathbb{Z}[\{a_{\mathbf{i}}\}]$ determined up to a unit, i.e., up to sign; Δ is called the discriminant. (See [GKZ08, Dem12, Sai12] for other descriptions of Δ.) By definition, if the $a_{\mathbf{i}}$ are specialized to elements of a field k, the resulting hypersurface in $\mathbb{P}^n_k$ is singular (not smooth of dimension $n-1$) if and only if Δ specializes to 0 in k.

3. Quadratic forms

Proposition 3.1. *Suppose that $d = 2$. Let $\operatorname{Det} = \det(\partial^2 F/\partial x_i \partial x_j) \in \mathbb{Z}[\{a_{\mathbf{i}}\}]$. If n is odd, then $\Delta = \pm \operatorname{Det}$. If n is even, then $\Delta = \pm \operatorname{Det}/2$.*

Proof. This is well known, except perhaps the power of 2, which can be determined by evaluating Det for a quadratic form defining a smooth quadric over $\mathbb{Z}$, since $\Delta = \pm 1$ for such a form. Use $x_0x_1 + \cdots + x_{n-1}x_n$ if n is odd, and $x_0x_1 + \cdots + x_{n-2}x_{n-1} + x_n^2$ if n is even. ☐

A symmetric bilinear space over R is a pair (M, β) where M is a finite-rank projective module R (hence free since R is a discrete valuation ring) and $\beta \colon M \times M \to R$ is a symmetric R-bilinear pairing.

Proposition 3.2. *Let R be a discrete valuation ring.*

(a) *Each symmetric bilinear space over R is an orthogonal direct sum of spaces of rank 1 and 2.*
(b) *Every quadratic form $f(x_0, \ldots, x_n)$ over R is equivalent to one of the form*

$$\sum_{i=1}^{I} (a_i x_i^2 + b_i x_i y_i + c_i y_i^2) + \sum_{j=1}^{J} d_j z_j^2$$

with $2I + J = n+1$ and $a_i, b_i, c_i, d_j \in R$.
(c) *Let f be as in* (b). *Let $H = \operatorname{Proj} R[x_0, \ldots, x_n]/(f)$. Then $v(\Delta(f)) \geq \dim (H_k)_{\text{sing}} + 1$.*

Proof.

(a) (We paraphrase an argument of Jean-Pierre Tignol adapted from the proof of [Ver19, Proposition 4.10].) Let (M, β) be a nonzero symmetric bilinear space. We may assume that $\beta \neq 0$. By dividing β by a nonzero element of R, we may assume that $\beta(M, M) \not\subseteq \mathfrak{m}$. We claim that there exists a free R-module N of rank 1 or 2 with a homomorphism $N \to M$ such that β induces a regular pairing on N (i.e., the composition $N \to M \xrightarrow{\beta} M^\vee \to N^\vee$ is an isomorphism); then $N \to M$ is injective, and M is the orthogonal direct sum of N and $N^\perp := \ker(M \to N^\vee)$, so we are done by induction on $\operatorname{rank}(M)$.

If there exists $e \in M$ with $\beta(e,e) \in R^\times$ a unit, then let $N = Re$. Otherwise, choose $c, d \in M$ with $\beta(c,d) \in R^\times$ and let $N = Rc \oplus Rd$; the induced pairing is regular since its matrix is invertible, being congruent mod π to $\begin{pmatrix} 0 & \beta(c,d) \\ \beta(c,d) & 0 \end{pmatrix}$.

(b) Decomposing a quadratic space is equivalent to decomposing the associated symmetric bilinear space, even if $\operatorname{char} k = 2$.

(c) First suppose $\operatorname{char} k \neq 2$. Then f is equivalent to $\sum a_i x_i^2$ for some $a_i \in R$, and

$$\dim (H_k)_{\mathrm{sing}} = \#\{i : v(a_i) \geq 1\} - 1 \leq v(\operatorname{Det}(f)) - 1 = v(\Delta(f)) - 1,$$

by Proposition 3.1.

Now suppose $\operatorname{char} k = 2$. Let $I_0 = \#\{i : v(b_i) = 0\}$ and $I_1 = \#\{i : v(b_i) \geq 1\}$. Let $J_0 = \#\{j : v(d_j) = 0\}$ and $J_1 = \#\{j : v(d_j) \geq 1\}$. If n is odd, let $J' := J$. If n is even, then J is odd, so let $J' := J - 1$. In both cases $J' \geq 0$. The common zero locus in $\mathbb{P}^n_k$ of the polynomials $\partial f / \partial x_i$ and $\partial f / \partial y_i$ for $i \in I_0$ is of dimension $n - 2I_0$, and including the condition $f = 0$ drops the dimension by 1 more if $J_0 \geq 1$. Thus $\dim (H_k)_{\mathrm{sing}} \leq n - 2I_0$, with strict inequality if $J_0 \geq 1$. On the other hand, $v(4a_i c_i - b_i^2) \geq 2$ whenever $v(b_i) \geq 1$, and $v(2d_j) \geq v(2) + v(d_j)$ for all j, so Proposition 3.1 implies

$$\begin{aligned} v(\Delta(f)) &\geq 2I_1 + J'v(2) + J_1 \\ &= (n - 2I_0) + J'v(2) - J_0 + 1 \\ &\geq \dim (H_k)_{\mathrm{sing}} + J'v(2) - J_0 + 1. \end{aligned}$$

If $J_0 \geq 1$, then the inequality above is strict and $J'v(2) \geq (J_0 - 1)v(2) \geq J_0 - 1$, so $v(\Delta(f)) \geq \dim (H_k)_{\mathrm{sing}} + 1$. If $J_0 = 0$, then instead use $J'v(2) \geq 0$ to again get $v(\Delta(f)) \geq \dim (H_k)_{\mathrm{sing}} + 1$. $\square$

4. Nondegenerate double points

Definition 4.1 ([SGA $\mathbf{7}_{\mathrm{I}}$, VI.6])**.** Let k be a field. Let X be a finite-type k-scheme. A k-point $Q \in X$ is called a nondegenerate double point (or nondegenerate quadratic point) if there exist $n \geq 1$ and $f \in k[[x_1, \ldots, x_n]]$ such that there is an isomorphism of complete k-algebras $\widehat{\mathscr{O}}_{X,Q} \simeq k[[x_1, \ldots, x_n]]/(f)$ and an equality of ideals $(\partial f/\partial x_1, \ldots, \partial f/\partial x_n) = (x_1, \ldots, x_n)$.

Remark 4.2. The ideal equality is equivalent to saying that Q is an isolated reduced point of the singular subscheme X_{sing}.

Remark 4.3. Suppose that n and f exist. Then f can be taken to be a quadratic form [SGA 7_{I}, VI.6.1]. If, moreover, k is algebraically closed, then

- if $\operatorname{char} k \neq 2$, then one can take $f := x_1^2 + \ldots + x_n^2$;
- if $\operatorname{char} k = 2$, then n must be even and one can take $f := x_1 x_2 + x_3 x_4 + \cdots + x_{n-1} x_n$.

Remark 4.4 ([SGA 7_{I}, Definition VI.6.6]). There is also notion of ordinary double point, which is the same except that when $\operatorname{char} k = 2$ and n is odd, since nondegeneracy is impossible one allows singularities analytically equivalent over an algebraic closure to the singularity defined by the "least degenerate" quadratic form $f := x_1 x_2 + \cdots + x_{n-2} x_{n-1} + x_n^2$.

5. Commutative algebra

A ring extension $R' \supset R$ is called a weakly unramified extension if R' too is a discrete valuation ring and π is also a uniformizer of R'.

Lemma 5.1. *For any field extension $k' \supset k$, there exists a weakly unramified extension $R' \supset R$ with residue field k' (i.e., isomorphic to k' as k-algebra).*

Proof. If k'/k is generated by one algebraic element, say a zero of a monic irreducible polynomial $\bar{f} \in k[x]$, then we may take $R' := R[x]/(f)$ for any monic $f \in R[x]$ reducing to $\bar{f}$ [Ser79, I.§6, Proposition 15]. If k'/k is generated by one transcendental element t, then we may take the localization $R' := R[t]_{(\pi)}$ of the (regular) polynomial ring $R[t]$ at the codimension 1 prime (π); the residue field of R' is $\operatorname{Frac}(R[t]/(\pi)) = k(t)$. The general case follows from Zorn's lemma, using direct limits. $\square$

Lemma 5.2. *Let A be a noetherian local domain. Let $\widehat{A}$ be its completion. Let B be the integral closure of A. Then*

$$\#\{\text{minimal primes of } \widehat{A}\} \geq \{\text{maximal ideals of } B\}.$$

Proof. Combine [SP, Tag 0C24] and [SP, Tag 0C28(1)]. $\square$

6. Hypersurfaces with several singularities

Let notation be as in Theorem 1.1. We use subscripts to denote base change: e.g., $D_A := D \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A$ for any ring A. Restricting ϕ_R yields a proper morphism $\varphi\colon (\mathcal{H}_R)_{\text{sing}} \to D_R$.

Proposition 6.1. *The proper morphism $\varphi\colon (\mathcal{H}_R)_{\text{sing}} \to D_R$ is birational.*

Proof. This follows from [Sai12, Proposition 2.12] applied over $\operatorname{Frac}(R)$. $\square$

Theorem 6.2. *If the space $(H_k)_{\text{sing}}$ consists of r closed points, then $v(\Delta(f)) \geq r$.*

Proof. Using Lemma 5.1, we may reduce to the case in which k is algebraically closed.

Let $P \in D_R(k)$ correspond to H_k, so $\varphi^{-1}(P) = (H_k)_{\text{sing}}$. Since R is regular, the local ring $\mathscr{O}_{\mathbb{A}^N_R,P}$ is regular, and hence factorial [AB59, Theorem 5].

Let $D' := \{d \in D_R : \dim \varphi^{-1}(d) = 0\}$, so $P \in D'$. By [EGA IV$_3$, Corollaire 13.1.5], D' is open in D_R. By Proposition 6.1, $\varphi^{-1}(D') \to D'$ is birational. It is also quasi-finite and proper, hence finite by Zariski's main theorem [EGA III$_1$, Corollaire 4.4.11]. Moreover, $(\mathcal{H}_R)_{\text{sing}}$ is smooth over a discrete valuation ring, hence normal. The previous three sentences imply that $\varphi^{-1}(D') \to D'$ is the normalization of D'.

Take $A := \mathscr{O}_{D',P} = \mathscr{O}_{D,P} = \mathscr{O}_{\mathbb{A}^N_R,P}/(\Delta)$, and define $\widehat{A}$ and B as in Lemma 5.2. Then the maximal ideals of B correspond to the points of $\varphi^{-1}(D')$ above P, which are the r points of $(H_k)_{\text{sing}}$. Lemma 5.2 implies that $\widehat{A}$ has at least r minimal primes. Their inverse images in $\mathscr{O}_{\mathbb{A}^N_R,P}$, correspond to prime factors of Δ in this factorial ring, so $\Delta = p_1 \cdots p_r q$, for some $p_1, \ldots, p_r, q \in \mathscr{O}_{\mathbb{A}^N_R,P}$ with each p_i vanishing at P. Evaluating both sides at (the coefficient tuple of) f shows that $v(\Delta(f)) \geq 1 + \cdots + 1 + 0 = r$. $\square$

7. Valuations of polynomial values

Lemma 7.1. *Suppose that k is infinite, and $\ell \geq n$. Let $\rho \colon \mathbb{A}^\ell_k \to \mathbb{A}^n_k$ be a projection. Let $V \subset \mathbb{A}^\ell_k$ be a closed subscheme. Then $\{a \in k^n : \rho^{-1}(a)(k) \subseteq V(k)\}$ is the set of k-points of a closed subscheme $Z \subseteq \mathbb{A}^n_k$.*

Proof. Since k is infinite, $\rho^{-1}(a)(k) \subset V(k)$ is equivalent to $\rho^{-1}(a) \subset V$, which *fails* if and only if $a \in \rho(\mathbb{A}^n_k - V)$. Since ρ is flat, ρ is open, so $\rho(\mathbb{A}^n_k - V)$ is open; let Z be its complement. □

For $b \in R$, let $\bar{b}$ be its image in k. Likewise, given $b \in R^n$, define $\bar{b} \in k^n$.

Proposition 7.2. *Let $\delta \in R[x_1, \ldots, x_n]$ and $m \in \mathbb{Z}_{\geq 0}$. If k is infinite and perfect, then*

$$\{a \in k^n : v(\delta(b)) \geq m \text{ for all } b \in R^n \text{ with } \bar{b} = a\}$$

is the set of k-points of a closed subscheme of $\mathbb{A}^n_k$.

Proof. The mth Greenberg functor Gr^m satisfies $\mathrm{Gr}^m(X)(k) = X(R/\mathfrak{m}^m)$ for any R-scheme X; see [Gre61; Gre63; NS08, §2.2; BGA18]. Applying Gr_m to $\delta \colon \mathbb{A}^n_R \to \mathbb{A}^1_R$ yields a morphism

$$\mathrm{Gr}^m(\mathbb{A}^n_R) \longrightarrow \mathrm{Gr}^m(\mathbb{A}^1_R);$$

let V be the fiber above 0. On the other hand, the reduction map $R/\mathfrak{m}^m \to k$ induces a morphism $\rho \colon \mathrm{Gr}^m(\mathbb{A}^n_R) \to \mathrm{Gr}^1(\mathbb{A}^n_R)$ that is a projection $\mathbb{A}^{mn}_k \to \mathbb{A}^n_k$ as in Lemma 7.1. For $a \in k^n$,

$$v(\delta(b)) \geq m \text{ for all } b \in R^n \text{ with } \bar{b} = a \iff \rho^{-1}(a)(k) \subset V(k),$$

so the result follows from Lemma 7.1. □

8. Hypersurfaces with a positive-dimensional singularity

In Lemma 8.1, Corollary 8.2, and Lemma 8.3, we assume that $n \geq 2$, $r \geq 1$, and $P_1, \ldots, P_r$ are distinct points in $\mathbb{P}^n(k)$. Let $\mathscr{O} = \mathscr{O}_{\mathbb{P}^n_k}$. For each $P \in \mathbb{P}^n(k)$, let $\mathfrak{m}_P \subset \mathscr{O}$ be the ideal sheaf of P.

Lemma 8.1. *If $d \geq 2r - 1$, then $\mathscr{O}(d) \to \prod_i (\mathscr{O}/\mathfrak{m}^2_{P_i})(d)$ induces a surjection on global sections.*

Proof. Let ℓ_i be a linear form vanishing at P_i but not P_j for any $j \neq i$. Let h be a homogeneous polynomial of degree $d - (2r - 1)$ not vanishing at any P_i. For each s, as g ranges over linear forms, the image of g in $(\mathscr{O}/\mathfrak{m}^2_{P_s})(1)$ ranges over all its sections, so the images of $gh \prod_{j \neq s} \ell_j^2$ in $\prod_i (\mathscr{O}/\mathfrak{m}^2_{P_i})(d)$ exhaust the sth factor of $\prod_i (\mathscr{O}/\mathfrak{m}^2_{P_i})(d)$. □

Corollary 8.2. *Let $N = \dim_k \Gamma(\mathbb{P}^n, \mathscr{O}(d))$. For $f \in \Gamma(\mathbb{P}^n, \mathscr{O}(d))$, let $H_f :=$* $\operatorname{Proj} k[x_0, \ldots, x_n]/(f)$. *Then the f for which $(H_f)_{\mathrm{sing}} \supset \{P_1, \ldots, P_r\}$ form a vector space of dimension $N - r(n+1)$.*

Lemma 8.3. *If $d \geq 3$ and $1 \leq r \leq \max((d-1)/2, 2)$, then in the locus $\mathbb{A}$ of f for which $(H_f)_{\mathrm{sing}} \supset \{P_1, \ldots, P_r\}$, the open sublocus U for which $(H_f)_{\mathrm{sing}}$ is finite is dense.*

Proof. Since $\mathbb{A}$ is defined by the vanishing of values of f and its partial derivatives at the P_i, it is cut out by linear forms in the coefficients of f, so $\mathbb{A}$ is an affine space. Applying [EGA IV_3, Corollaire 13.1.5] the relative singular subscheme over $\mathbb{A}$ shows that U is open in $\mathbb{A}$, so it remains to show that $U \neq \emptyset$.

First suppose that $r \leq (d-1)/2$. Let

$$I = \{(f, P_{r+1}) : f \in \mathbb{A},\ P_{r+1} \in (H_f)_{\mathrm{sing}} - \{P_1, \ldots, P_r\}\}.$$

The fiber of $I \to \mathbb{P}^n_k - \{P_1, \ldots, P_r\}$ above P_{r+1} consists of the f for which $(H_f)_{\text{sing}} \supset \{P_1, \ldots, P_{r+1}\}$, so its dimension is $N - (r+1)(n+1)$ by Corollary 8.2; similarly, $\dim \mathbb{A} = N - r(n+1)$. Thus $\dim I = n + N - (r+1)(n+1) = \dim \mathbb{A} - 1$. Therefore $I \to \mathbb{A}$ is not dominant, and U contains the complement of its image.

Now suppose instead that $r \leq 2$. Choose a homogeneous degree d form $g(x_3, \ldots, x_n)$ defining a smooth hypersurface in $\mathbb{P}^{n-3}$, let $c_1, \ldots, c_{d-1} \in k$ be distinct (enlarge k if necessary), and let

$$f = x_0 \prod_{i=1}^{d-1} (x_1 - c_i x_2) + g.$$

At a point P where f and its partial derivatives vanish, $\prod_{i=1}^{d-1}(x_1 - c_i x_2) = 0$, so $g = 0$, so g and its derivatives vanish, so $x_3 = \cdots = x_n = 0$; thus P is a singular point of the plane curve $x_0 \prod_{i=1}^{d-1}(x_1 - c_i x_2) = 0$, i.e., an intersection point of two components. By a linear change of variable, we may assume that the P_i (of which there are at most two!) are among these singular points. Then f gives a k-point of U. □

Theorem 8.4. *Let $H = \operatorname{Proj} R[x_0, \ldots, x_n]/(f)$ for some homogeneous f of degree d. If $\dim (H_k)_{\text{sing}} \geq 1$, then $v(\Delta(f)) \geq \max(\lfloor (d-1)/2 \rfloor, 2)$.*

Proof. We may assume that $n, d \geq 2$. Using Lemma 5.1, we may reduce to the case in which k is algebraically closed. If $d = 2$, then Proposition 3.2(c) implies that $v(\Delta(f)) \geq \dim (H_k)_{\text{sing}} + 1 \geq 2$.

So assume $d \geq 3$. Let Z be the closed subscheme of Proposition 7.2 for $\delta := \Delta \in R[\{a_{\mathbf{i}}\}]$ and $r := \max(\lfloor (d-1)/2 \rfloor, 2)$. Choose distinct $P_1, \ldots, P_r \in (H_k)_{\text{sing}}(k)$. If $j \in R[x_0, \ldots, x_n]$ is a degree d homogeneous polynomial, and $J = \operatorname{Proj} R[x_0, \ldots, x_n]/(j)$ is such that $(J_k)_{\text{sing}} = \{P_1, \ldots, P_r\}$, then $v(\Delta(j)) \geq r$ by Theorem 6.2, so the corresponding coefficient tuple mod $\mathfrak{m}$ belongs to $Z(k)$. By Lemma 8.3, any coefficient tuple mod $\mathfrak{m}$ corresponding to a hypersurface whose singular locus contains $\{P_1, \ldots, P_r\}$ also belongs to $Z(k)$. This applies in particular to the coefficient tuple of f mod $\mathfrak{m}$, so $v(\Delta(m)) \geq r$ by definition of Z. □

9. When the discriminant has valuation 1

Proof of Theorem 1.1. Case 1: $\operatorname{char} k = 2$ *and n is odd.* By [Sai12, Theorem 4.2], if the sign of Δ is chosen appropriately, then $\Delta = A^2 + 4B$ for some polynomials A, B, so $v(\Delta(f)) \neq 1$. On the other hand, by Remark 4.3, H_k cannot have a nondegenerate double point. Thus (i) and (ii) both fail.

Case 2: $\operatorname{char} k \neq 2$ *or n is even.* The hypersurface $H \to \operatorname{Spec} R$ is the pullback of $\mathcal{H}_R \to \mathbb{A}^N_R$ by some R-morphism $\iota\colon \operatorname{Spec} R \to \mathbb{A}^N_R$. Let $P = \iota(\operatorname{Spec} k) \in \mathbb{A}^N(k)$.

(i)⇒(ii): Suppose that $v(\Delta(f)) = 1$. By Theorem 8.4, $(H_k)_{\text{sing}}$ is finite. The surjection $R[\{a_{\mathbf{i}}\}] \twoheadrightarrow R$ sending the $a_{\mathbf{i}}$ to the corresponding coefficients $\alpha_{\mathbf{i}}$ of f maps Δ to $\Delta(f)$, so the $a_{\mathbf{i}} - \alpha_{\mathbf{i}}$ and Δ are local parameters for $\mathbb{A}^N_R$ at P. Thus $D_R = \operatorname{Spec} R[\{a_{\mathbf{i}}\}]/(\Delta)$ is regular at P, so D_R is normal at P. Let U be the largest normal open subscheme of D_R such that $\varphi^{-1}U \to U$ has finite fibers. The fiber above P is $(H_k)_{\text{sing}}$, so $P \in U$. By Proposition 6.1, φ is a proper birational morphism, so $\varphi^{-1}U \to U$ has finite fibers by Zariski's main theorem [EGA III_1, Corollaire 4.4.9]. In particular, the fiber $(H_k)_{\text{sing}}$ consists of a single reduced k-point Q. By Remark 4.2, Q is a nondegenerate double point of H_k.

Choose an $\mathbb{A}^n_R \subset \mathbb{P}^n_R$ containing Q; let f_0 be the corresponding dehomogenization of f. The point $(H_k)_{\text{sing}}$ is cut out in $\mathbb{A}^n_R$ by f_0 and its partial derivatives; these $n+1$ functions are therefore local parameters for $\mathbb{P}^n_R$ at Q, so the local ring $\mathscr{O}_{H,Q} = \mathscr{O}_{\mathbb{P}^n_R,Q}/(f_0)$ is regular too. On the other hand, $H - \{Q\}$ is smooth over $\operatorname{Spec} R$. Thus H is regular everywhere.

(ii)⇒(i): Now suppose that H is regular and $(H_k)_{\mathrm{sing}}$ consists of a nondegenerate double point $Q \in H(k)$. Hence the underlying space of H_{sing} is $\{Q\}$.

Since the tangent space of $(H_k)_{\mathrm{sing}}$ at Q is 0, the projection $(\mathcal{H}_k)_{\mathrm{sing}} \to \mathbb{A}^N_k$ induces an injection between the tangent spaces at Q and P. Since Q is the only point in $(\mathcal{H}_k)_{\mathrm{sing}}$ above P, this implies that $(\mathcal{H}_R)_{\mathrm{sing}} \to D_R$ is étale at Q. Pulling back $(\mathcal{H}_R)_{\mathrm{sing}} \to D_R \hookrightarrow \mathbb{A}^N_R$ by ι shows that $H_{\mathrm{sing}} \to \operatorname{Spec}(R/(\Delta(f)))$ is étale. These are connected 0-dimensional schemes with the same residue field, so $H_{\mathrm{sing}} \simeq \operatorname{Spec}(R/(\Delta(f)))$.

Let f_0 be as above, so f_0 and its partial derivatives lie in the maximal ideal $\mathfrak{m}_{\mathbb{P}^n_R,Q} \subset \mathscr{O}_{\mathbb{P}^n_R,Q}/(f_0)$. The partial derivatives are independent in $\mathfrak{m}_{\mathbb{P}^n_R,Q}/\mathfrak{m}^2_{\mathbb{P}^n_R,Q}$ since they form a basis for $\mathfrak{m}_{\mathbb{P}^n_k,Q}/\mathfrak{m}^2_{\mathbb{P}^n_k,Q}$, since Q is a nondegenerate double point. On the other hand, the image of f_0 in $\mathfrak{m}_{\mathbb{P}^n_R,Q}/\mathfrak{m}^2_{\mathbb{P}^n_R,Q}$ is nonzero (since $\mathscr{O}_{H,Q} = \mathscr{O}_{\mathbb{P}^n_R,Q}/(f_0)$ is regular) and in fact *independent* of the partial derivatives (since it maps to 0 in $\mathfrak{m}_{\mathbb{P}^n_k,Q}/\mathfrak{m}^2_{\mathbb{P}^n_k,Q}$). Thus f_0 and its partial derivatives form a basis of $\mathfrak{m}_{\mathbb{P}^n_R,Q}/\mathfrak{m}^2_{\mathbb{P}^n_R,Q}$, so by Nakayama's lemma, they generate $\mathfrak{m}_{\mathbb{P}^n_R,Q}$, so $H_{\mathrm{sing}} \simeq \operatorname{Spec} k$.

The conclusions of the two previous paragraphs imply $v(\Delta(f)) = 1$. ☐

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References

[AB59] Maurice Auslander and D. A. Buchsbaum, *Unique factorization in regular local rings*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 733–734, DOI 10.1073/pnas.45.5.733. MR103906 ↑4

[BGA18] Alessandra Bertapelle and Cristian D. González-Avilés, *The Greenberg functor revisited*, Eur. J. Math. **4** (2018), no. 4, 1340–1389, DOI 10.1007/s40879-017-0210-0. MR3866700 ↑5

[Dem12] Michel Demazure, *Résultant, discriminant*, Enseign. Math. (2) **58** (2012), no. 3-4, 333–373 (French). MR3058604 ↑2

[EGA III_1] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. **11** (1961). Written in collaboration with J. Dieudonné. MR0217085 (36 #177c) ↑4, 6

[EGA IV_3] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. **28** (1966). Written in collaboration with J. Dieudonné. MR0217086 (36 #178) ↑4, 5

[GKZ08] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Modern Birkhäuser Classics, Birkhäuser Boston Inc., Boston, MA, 2008. Reprint of the 1994 edition. MR2394437 (2009a:14065) ↑2

[Gre61] Marvin J. Greenberg, *Schemata over local rings*, Ann. of Math. (2) **73** (1961), 624–648, DOI 10.2307/1970321. MR126449 ↑5

[Gre63] Marvin J. Greenberg, *Schemata over local rings. II*, Ann. of Math. (2) **78** (1963), 256–266, DOI 10.2307/1970342. MR156855 ↑5

[NS08] Johannes Nicaise and Julien Sebag, *Motivic Serre invariants and Weil restriction*, J. Algebra **319** (2008), no. 4, 1585–1610, DOI 10.1016/j.jalgebra.2007.11.006. MR2383059 ↑5

[Sai12] Takeshi Saito, *The discriminant and the determinant of a hypersurface of even dimension*, Math. Res. Lett. **19** (2012), no. 4, 855–871, DOI 10.4310/MRL.2012.v19.n4.a10. MR3008420 ↑1, 2, 4, 6

[Ser79] Jean-Pierre Serre, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg. MR554237 (82e:12016) ↑4

[SGA $7_{\rm I}$] *Groupes de monodromie en géométrie algébrique. I*, Lecture Notes in Mathematics, vol. 288, Springer-Verlag, Berlin, 1972 (French). Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I); Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim. MR0354656 (50 #7134) ↑3

[Sil94] Joseph H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer-Verlag, New York, 1994. MR96b:11074 ↑1

[SP] The Stacks Project authors, *Stacks project*, May 18, 2020. Available at `http://stacks.math.columbia.edu` . ↑4

[Tat75] J. Tate, *Algorithm for determining the type of a singular fiber in an elliptic pencil*, Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Springer, Berlin, 1975, pp. 33–52. Lecture Notes in Math., Vol. 476. MR0393039 ↑1

[Ver19] Joachim Verstraete, *Arason's filtration of the Witt group of dyadic valued fields*, J. Algebra **519** (2019), 190–227, DOI 10.1016/j.jalgebra.2018.10.026. MR3876186 ↑2

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