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ALGEBRAIC GEOMETRY

Remarks on F-spans

by

Arthur OGUS

Presented by Wiesława NIZIOŁ

Summary. We prove a version of Frobenius descent with applications to the theory of F-crystals and F-spans. Let X/S be a smooth morphism of schemes in characteristic p, let $F_{X/S}\colon X/S \to X'/S$ be the relative Frobenius morphism, and let $F\colon Z/T \to Z'/T$ be a lifting of $F_{X/S}$, where Z, Z', and T are p-torsion free p-adic formal schemes. If (E', ∇') is a p-torsion free coherent sheaf with integrable and quasi-nilpotent connection on Z', we show that any submodule of $F^*(E')$ which is invariant under the induced connection descends to E'. As a consequence, we show that if $\Phi\colon F^*(E', \nabla') \to (E, \nabla)$ is an F-span, then the Mazur–Nygaard filtration on $F^*_{X/S}(E'_{X'})$ descends naturally to a filtration on $E'_{X'}$ which satisfies Griffiths transversality. This generalizes an earlier result of the author, which required that the Frobenius lift F be of a special form. We also investigate how the Mazur–Nygaard filtration depends on the lifting F.

Let k be a field of characteristic $p > 0$ and W its Witt ring. Mazur's fundamental work [2] (slightly generalized in [1]) showed that if Y/k is a smooth projective scheme, then, under a mild hypothesis, the Hodge filtration Fil of $H^\bullet_{\mathrm{DR}}(Y/k)$ agrees with the "abstract" Hodge filtration defined by the Frobenius endomorphism Φ of $H^\bullet_{\mathrm{cris}}(Y/W)$. Namely, $\mathrm{Fil}^i\, H^\bullet_{\mathrm{DR}}(Y/k)$ is the image of $\Phi^{-1}(p^i H^\bullet_{\mathrm{cris}}(Y/W))$ under the canonical map

$$H^\bullet_{\mathrm{cris}}(Y/W) \to H^\bullet_{\mathrm{cris}}(Y/k) \cong H^\bullet_{\mathrm{DR}}(Y/k).$$

Inspired by Mazur's result, the current author attempted to formulate an analog in which k is replaced by a smooth scheme S/k and in which Y/k is replaced by a smooth and proper Y/S. The formalism of crystalline cohomology then produces an "F-crystal" on S. Concretely this means that,

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for each p-adic formal scheme T/W lifting S, we get an $\mathcal{O}_T$-module with integrable connection (E, ∇), and, for each lifting F_T of F_S, a horizontal morphism

$$\tilde{\Phi}\colon F_T^*(E,\nabla) \to (E,\nabla).$$

Then one can follow Mazur to define a filtration on $F_T^*(E,\nabla)$ by

$$M^i F_T^*(E) := \tilde{\Phi}^{-1}(p^i E)$$

and then look at its image in $F_S^*(E/pE)$. However, this filtration is horizontal, and the analog of the Hodge filtration we seek lives in E/pE rather than $F_S^*(E/pE)$ and should satisfy Griffiths transversality rather than horizontality. To obtain it, one must show that the filtration $M^\bullet$ above descends to a filtration $A^\bullet$ of E and that $A^\bullet$ is Griffiths transversal. In [3], this is shown to hold assuming that the Frobenius lift F_T satisfies a hypothesis: there should exist local coordinates $(t_1, \ldots, t_r)$ such that $F_T^*(t_i) = t_i^p$ for all i. Recently K. Kato has pointed out that this hypothesis seems to be superfluous. This note is my attempt to understand his comment, for which I am grateful. I would also like to thank the referee for a careful reading of the first submission of this manuscript and for some helpful suggestions about the exposition.

To state our result, we change the setup slightly. Let X/S be a smooth morphism of schemes in characteristic p, and let $F_{X/S}\colon X \to X'$ be the relative Frobenius morphism. Suppose that T is a p-torsion free p-adic formal scheme whose reduction modulo p is S and that $F\colon Z \to Z'$ is a lifting of $F_{X/S}$, where Z/T and Z'/T are formally smooth formal liftings of X/S and X'/S respectively. (All schemes and formal schemes will be assumed to be noetherian.)

Our main goal is the following result.

THEOREM 1. *Let (E', ∇') be a p-torsion free coherent sheaf with integrable and quasi-nilpotent connection on Z'/T. Suppose that E is a submodule of F^*E' which is invariant under the induced connection on F^*E'. Let $\eta\colon E' \to F_*F^*E'$ be the adjunction map. Then the natural map*

$$F^*(\eta^{-1}(F_*E)) \to E$$

is an isomorphism.

Since F is faithfully flat, the most natural way to attack this problem would be to show that E is necessarily invariant under the descent data for F^*E'. In fact, as we explain later, the result can be deduced from Shiho's Theorem 3.1 in [5] which uses related, but different, descent data. Here we follow a different approach, based on Cartier descent, working in the context of F-crystals, which in fact is where the above question found its origin.

We recall some notions from [3]. Let $F\colon Z \to Z'$ be a lifting of $F_{X/S}$ as above. Then an *F-span on X/S* is given by a pair of p-torsion free $\mathcal{O}_Z$-

modules with quasi-nilpotent integrable connection (E', ∇') on Z'/W and (E, ∇) on Z/W together with an injective homomorphism

$$\tilde{\Phi}\colon F^*(E', \nabla') \to (E, \nabla)$$

whose cokernel is killed by p^n for some $n > 0$. (Note that this is less data than that of an F-span, since now the source and target of $\tilde{\Phi}$ are decoupled.)

We have a commutative diagram:

$$\begin{array}{ccc} E' & \xrightarrow{\ \eta\ } & F_*F^*E' \\ & \searrow^{\Phi} & \downarrow{\scriptstyle F_*\tilde{\Phi}} \\ & & F_*E \end{array}$$

These maps and the filtrations they lead to depend on the choice of the Frobenius lift, and when we feel it necessary we will indicate this by adding a subscript.

Definition 2. If $\tilde{\Phi}\colon F^*(E', \nabla') \to (E, \nabla)$ is an F-span, let

$$\begin{aligned} M^iF^*E' &:= \tilde{\Phi}^{-1}(p^iE), \\ A^iE' &:= \Phi^{-1}(p^iE) = \eta^{-1}(M^iF^*E'), \\ M^{[i]}F^*E' &:= \sum_j p^{[j]}M^{i-j}F^*E', \\ A^{[i]}E' &:= \sum_j p^{[j]}A^{i-j}E'. \end{aligned}$$

We shall show in Proposition 8 that the filtration $A^{[\bullet]}$, unlike $A^{\bullet}$, is independent of the lifting F, allowing a conceptual simplification of some of the constructions of [4].

In what follows we shall often write $(\tilde{E}, \tilde{\nabla})$ for $F^*(E', \nabla')$. Note that $E/pE = E_X$, the restriction of E to X, and similarly for E' and $\tilde{E}$. The map $p^{-i}\tilde{\Phi}$ induces a morphism $M^i\tilde{E} \to E$; we denote by $N^{-i}E$ its image. Thus we find an isomorphism

$$\tilde{\Phi}_i\colon M^i\tilde{E} \to N^{-i}E.$$

The proof of the following proposition relating the associated graded modules of these filtrations is a straightforward consequence of the definitions and is omitted.

Proposition 3. *The filtrations $M^{\cdot}\tilde{E}$, $M^{[\bullet]}\tilde{E}$ are stable under $\tilde{\nabla}$, and the filtration $N^{\bullet}E$ is stable under ∇. The filtrations $A^{\bullet}E'$ and $A^{[\bullet]}E'$ satisfy Griffiths transversality. The isomorphisms $\tilde{\Phi}_i\colon M^i\tilde{E} \to N^{-i}E$ are compatible* *with the connections, and induce isomorphisms*

$$(\mathrm{gr}^i_M \tilde{E}, \tilde{\nabla}) \to (N^{-i} E_X, \nabla),$$
$$(\mathrm{gr}^i_M \tilde{E}_X, \tilde{\nabla}) \to (\mathrm{gr}^{-i}_N E_X, \nabla).$$

THEOREM 4. *With the notation above, and for every i, the following statements hold:*

(1) *The natural map $F^*(A^i E') \to M^i \tilde{E}$ is an isomorphism.*
(2) *The natural maps $F^*(A^i E'_X) \to M^i \tilde{E}_X$ and $A^i E'_X \to (M^i E_X)^{\nabla}$ are isomorphisms.*
(3) *The natural maps $F^* \mathrm{gr}^i_A E'_X \to \mathrm{gr}^i_M \tilde{E}_X$ and $\mathrm{gr}^i_A E'_X \to (\mathrm{gr}^i_M \tilde{E}_X)^{\tilde{\nabla}}$ are isomorphisms.*
(4) *The natural map $F^* \mathrm{gr}^i_A E' \to \mathrm{gr}^i_M \tilde{E}$ is an isomorphism.*

Proof. We prove these statements together by induction on i. Suppose $i = 0$. Then statement (1) is true by definition, and it follows that $F^* E'_X \cong \tilde{E}_X$. Cartier descent implies that $E'_X = (F^* E'_X)^{\tilde{\nabla}}$. This proves (2).

Since $A^1 E' = \eta^{-1}(M^1 \tilde{E})$ and contains pE', it follows that

$$A^1 E' = E' \times_{\tilde{E}_X} M^1 \tilde{E}_X,$$

and hence

$$A^1 E'_X = E'_X \times_{\tilde{E}_X} M^1 \tilde{E}_X.$$

Since $E'_X \times_{\tilde{E}_X} M^1 \tilde{E}_X = (M^1 \tilde{E}_X)^{\nabla}$ and since the p-curvature of $M^1 \tilde{E}_X$ vanishes, it follows by Cartier descent that the natural map $F^*(A^1 E'_X) \to M^1 \tilde{E}_X$ is an isomorphism. Then statement (2) for $i = 0$ implies that the map $F^* \mathrm{gr}^0_A E' \to \mathrm{gr}^0_M \tilde{E}$ is an isomorphism. By Cartier descent again, it follows that the map $\mathrm{gr}^0_A E' \to (\mathrm{gr}^0_M \tilde{E})^{\tilde{\nabla}}$ is an isomorphism. This proves (3). Since $\mathrm{gr}^0_A E' = \mathrm{gr}^0_A E'_X$ and $\mathrm{gr}^0_M \tilde{E} = \mathrm{gr}^0_M \tilde{E}_X$, (4) follows as well.

For the induction step, assume that the statements hold for all $j < i$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^* A^i E' & \longrightarrow & F^* A^{i-1} E' & \longrightarrow & F^* \mathrm{gr}^{i-1}_A E' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^i \tilde{E} & \longrightarrow & M^{i-1} \tilde{E} & \longrightarrow & \mathrm{gr}^{i-1}_M \tilde{E} & \longrightarrow & 0
\end{array}$$

Statement (1) for $i - 1$ implies that the middle vertical arrow is an isomorphism and statement (4) for $i - 1$ implies that the right vertical arrow is an isomorphism. Thus the left vertical arrow is also an isomorphism, proving statement (1) for i.

We also have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^*A^iE'_X & \longrightarrow & F^*A^{i-1}E'_X & \longrightarrow & F^*\mathrm{gr}_A^{i-1}E'_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M^i\tilde{E}_X & \longrightarrow & M^{i-1}\tilde{E}_X & \longrightarrow & \mathrm{gr}_M^{i-1}\tilde{E}_X & \longrightarrow & 0 \end{array}$$

Statements (2) and (3) for $i-1$ imply that the two vertical maps on the right are isomorphisms, and consequently so is the map on the left. Since the p-curvature of $M^i\tilde{E}_X$ vanishes, statement (2) holds for i.

The main difficulty is in the following lemma, which corresponds to [3, 2.2.1] and will allow us to prove statement (3).

LEMMA 5. *If the statements of Theorem 4 hold for all $j < i$, then the map $\mathrm{gr}_A^i E'_X \to F_*\mathrm{gr}_M^i\tilde{E}_X$ is injective.*

Proof. Suppose $a' \in A^iE'$ lifts an element of the kernel of the map in the lemma. Then $\eta(a') = pb + c$, with $c \in M^{i+1}\tilde{E}$ and $b \in \tilde{E}$. Suppose that in fact

$$\eta(a') = p^jb + c$$

with $c \in M^{i+1}\tilde{E}$ and $j > 0$. Since $\eta(a') \in M^iE$, it follows that $b \in M^{i-j}\tilde{E}$. Note that if $j > i$, in fact $a' \in A^{i+1}E'$ and we are done. On the other hand, if $0 < j \le i$, we calculate

$$\begin{aligned} \tilde{\Phi}(\eta(a')) &= p^j\tilde{\Phi}(b) + \tilde{\Phi}(c), \\ \tilde{\Phi}_i(\eta(a')) &= \tilde{\Phi}_{i-j}(b) + p\tilde{\Phi}_{i+1}(c), \\ \nabla\tilde{\Phi}_i(\eta(a')) &= \nabla\tilde{\Phi}_{i-j}(b) + p\nabla\tilde{\Phi}_{i+1}(c). \end{aligned}$$

Since $\nabla\tilde{\Phi}_i(\eta(a')) = \tilde{\Phi}_i(\tilde{\nabla}(\eta(a')))$ is divisible by p, the same is true of $\nabla\tilde{\Phi}_{i-j}(b)$. We saw in Proposition 3 that the map $\tilde{\Phi}_{i-j}$ induces a horizontal isomorphism $\mathrm{gr}_M^{i-j}\tilde{E}_X \cong \mathrm{gr}_N^{j-i}E_X$, and we conclude that the image of b in $\mathrm{gr}_M^{i-j}\tilde{E}_X$ is horizontal. Statement (3) for $i-j$ says that $\mathrm{gr}_A^{i-j}E'_X \cong (\mathrm{gr}_M^{i-j}\tilde{E}_X)^{\tilde{\nabla}}$, so there exist $b' \in A^{i-j}E'$, $b'' \in \tilde{E}$, and $b''' \in M^{i-j-1}\tilde{E}$ such that

$$b = \eta(b') + b'' + pb'''.$$

Then

$$\eta(a' - p^jb') = p^jb + c - p^j\eta(b') = p^jb'' + p^{j+1}b''' + c.$$

Since $p^jb'' \in M^{i+1}\tilde{E}$, we can set $a'' := a' - p^jb'$ and $c' = p^jb'' + c$, so now $\eta(a'') = p^{j+1}b''' + c'$. Continuing by induction, we see that eventually a' may be chosen to lie in $A^{i+1}E'_X$. ■

Lemma 5 implies that $A^{i+1}E'_X = A^iE'_X \times_{M^i\tilde{E}_X} M^{i+1}\tilde{E}_X$. Since $A^iE'_X = (M^i\tilde{E}_X)^{\tilde{\nabla}}$, it follows that $A^{i+1}E'_X = (M^{i+1}\tilde{E}_X)^{\tilde{\nabla}}$, and since the p-curvature of $\tilde{\nabla}$ on $M^{i+1}\tilde{E}_X$ vanishes, the map $F^*(A^{i+1}E'_X) \to M^{i+1}\tilde{E}_X$ is an isomorphism. Thus (2) holds for $i+1$ and (3) holds for i.

Finally, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^*\mathrm{gr}_A^{i-1}E' & \xrightarrow{[p]} & F^*\mathrm{gr}_A^i\tilde{E} & \longrightarrow & F^*\mathrm{gr}_A^iE'_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{gr}_M^{i-1}\tilde{E} & \xrightarrow{[p]} & \mathrm{gr}_M^i\tilde{E} & \longrightarrow & \mathrm{gr}_M^i\tilde{E}_X & \longrightarrow & 0 \end{array}$$

The left vertical arrow is an isomorphism by the induction assumption and the right vertical arrow is an isomorphism by (3). It follows that the middle arrow is an isomorphism, proving (4) and completing the proof of Theorem 4. ■

It is fairly straightforward to transport our study of F-spans to the situation of Theorem 1.

Proof of Theorem 1. First suppose that there is a natural number n such that $p^nF^*E' \subseteq E$. Then multiplication by p^n induces a horizontal map $\tilde{\Phi}\colon F^*E' \to E$, defining an F-span, and $E = M^nF^*E'$. Since $A^nE' = \eta^{-1}(M^nF^*E')$, statement (1) of Theorem 4 tells us that the natural map $F^*(\eta^{-1}(E)) \to E$ is an isomorphism.

For the general case, let $E_n := E + p^nF^*E'$ for each n, each of which is also invariant under $\tilde{\nabla}$, and let $E'_n := \eta^{-1}(E_n)$ and $E'' := \eta^{-1}(E)$. We claim that in fact

$$E'' = \cap\{E'_n : n \geq 0\}.$$

Indeed, $E'/E'' \subseteq (F^*E')/E$, and the Artin–Rees lemma implies that for some $r > 0$,

$$(p^{n+r}\tilde{E}/E) \cap (E'/E'') \subseteq p^nE'/E'''$$

for all $n \geq 0$. Then $E_{n+r} \cap E' \subseteq E'' + p^nE'$ for all $n \geq 0$. Taking the intersection over n, we find that

$$E'' \subseteq \bigcap\{E'_n : n \geq 0\} = \bigcap\{E' \cap E_n : n \geq 0\} \subseteq \bigcap\{E'' + p^nE' : n \geq 0\}.$$

This proves our claim, since E'' is is necessarily p-adically closed in E', by the coherence assumption.

Since F is finite and flat, the natural map

$$F^*E'' = F^*\Big(\bigcap\{E'_n : n \geq 0\}\Big) \to \bigcap\{F^*E'_n : n \geq 0\}$$

is an isomorphism. The previous paragraph tells us that each map $F^*E'_n \to E_n$ is an isomorphism, so we conclude that the map

$$F^*E'' \to \bigcap\{E_n : n \geq 0\}$$

is an isomorphism. Since E is aso p-adically closed, this proves the result. ■

Let us now review Shiho's Theorem 3.1 of [5] and explain how it implies Theorem 1. The theorem constructs an equivalence C_F from the category of modules with integrable nilpotent p-connection on Z'/T to the category of modules with integrable nilpotent connection on Z/T as follows. Recall that there is a unique map

$$\zeta_F \colon \Omega^1_{Z'/S} \to F_* \Omega^1_{Z/S}$$

such that $p\zeta_F$ is the differential of F. Then if (E', θ') is a module with p-connection on Z', it is easy to verify that there is a unique connection $\tilde{\nabla}$ on $\tilde{E} := F^*E'$ such that $\tilde{\nabla} \circ \eta = (\zeta_F \otimes \eta) \circ \theta'$. Shiho proves that the functor C_F taking (E', θ') to $(\tilde{E}, \tilde{\nabla})$ is an equivalence by studying the descent data for the PD-thickenings which correspond to the crystalline interpretations of these categories.

To apply Shiho's result, suppose that (E', ∇') is a module with integrable nilpotent connection on Z'/T. Then $\theta' := p\nabla'$ is a nilpotent p-connection on E', and

$$(\zeta_F \otimes \eta) \circ \theta' = (p\zeta_F \otimes \eta) \circ \nabla' = (F^* \otimes \eta) \circ \theta',$$

Thus, the connection $\tilde{\nabla}$ in Shiho's correspondence is just the Frobenius pull-back connection on $\tilde{E} := F^*(E')$. Let $E \subseteq \tilde{E}$ be a submodule which is invariant under $\tilde{\nabla}$. We claim that the induced connection on E is also nilpotent. To see this, let $N^iE := p^i\tilde{E} \cap E$, which is also invariant under $\tilde{\nabla}$, and note that the inclusion map induces an injection $\mathrm{gr}^i_N E \to p^i\tilde{E}/p^{i+1}\tilde{E} \cong \tilde{E}/p\tilde{E}$. It follows that the p-curvature of each $\mathrm{gr}^i_N E$ vanishes. On the other hand, we have a surjection $\mathrm{gr}^i_N E \to (N^iE + pE)/(N^{i+1} + pE)$, so the p-curvature of each of the latter also vanishes. By Artin–Rees, there is a natural number r such that $p^r\tilde{E} \cap E \subseteq pE$. Thus the images of N^iE in E/pE define a finite and exhaustive filtration of E/pE, and so the connection on E is indeed nilpotent. Then the full faithfulness of C_F implies that there is a submodule E'' of E', stable under $\theta' := p\nabla$, such that $F^*E'' = E \subseteq \tilde{E}$. Necessarily $E' \subseteq \eta^{-1}(E)$, and since $F^*E' \cong E$, it follows that $E' = \eta^{-1}(E)$, as claimed.

Let us gather some implications of these results for the filtrations in Definition 2 associated to an F-span.

Proposition 6. *If $\tilde{\Phi} \colon F^*E' \to E$ is an F-span, then*

$$F^*(A^iE') \to M^iF^*E',$$
$$F^*(A^{[i]}E') \to M^{[i]}F^*E',$$
$$A^iE' \to \eta^{-1}(M^iF^*E'),$$
$$A^{[i]}E' \to \eta^{-1}(M^{[i]}F^*E')$$

are isomorphisms.

Proof. The third assertion is true by definition, and statement (1) of Theorem 4 proves the first equation. Since F is flat, the natural maps $F^*A^i_F E' \to F^*E'$ and $F^*A^{[i]}_F E' \to F^*E'$ are injective, and moreover

$$F^*A^{[i]}_F E' = \sum_j p^{[i-j]} F^* A^{i-j}_F E' = M^{[i]}_F F^*E'.$$

This is the second case of the proposition, which implies the fourth, by the lemma below. ∎

Lemma 7. *If A is a coherent $\mathcal{O}_{X'}$-submodule of E' and M is the image of F^*A in F^*E', then $A = \eta_F^{-1}(F_*M)$.*

Proof. Since F is flat, the map $F^*A \to F^*E'$ is injective. Let $A' := \eta_F^{-1}(M)$. Then A' is an $\mathcal{O}_{X'}$-submodule of E' and $A \subseteq A'$. We have injections $F^*A \to F^*A' \to M$ whose composition is an isomorphism. Then the map $F^*A \to F^*A'$ is an isomorphism, and since F is faithfully flat, $A = A'$. ∎

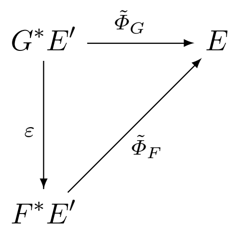
Let us now discuss how these filtrations vary with the choice of a Frobenius lift. If G is another lifting of $F_{X/S}$, the connection ∇' furnishes a horizontal isomorphism

$$\varepsilon_{G,F} \colon G^*(E', \nabla') \to F^*(E', \nabla').$$

and hence a map

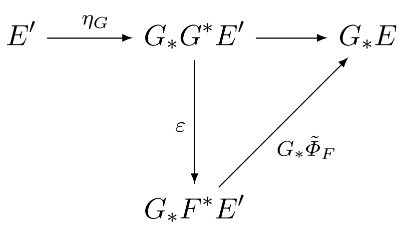
$$\tilde{\Phi}_G \colon G^*(E', \nabla') \to (E, \nabla),$$

making the diagram



commutative. It follows that the isomorphism ε takes $M^i_G G^*E'$ isomorphically to $M^i_F F^*E'$, and that $N^{-i}_F E = N^{-i}_G E$.

The diagram



shows that

$$\Phi_G = G_*(\tilde{\Phi}_F) \circ \varepsilon_{G,F}. \tag{1}$$

However, it need not be the case that $A^i_F E' = A^i_G E'$; we give an example below. The following proposition partially remedies this situation.

PROPOSITION 8. *Let F and G be liftings of $F_{X/S}$ to maps $Z \to Z'$. Then $A^{[i]}_F E' = A^{[i]}_G E'$ for all i.*

Proof. For simplicity we write the rest of the proof assuming that X is a curve and that it admits a local coordinate t. Let $\partial := \nabla'(d/dt)$ and suppose that $G^*(t) = F^*(t) + pg$. Then if $e' \in E'$,

$$\varepsilon(\eta_G(e')) = \sum_j p^{[j]} g^j \eta_F(\partial^j(e')). \tag{2}$$

Note that since the connection ∇' is nilpotent, the sequence $\partial^j(e')$ converges to zero.

To prove that $A^{[i]}_G E' \subseteq A^{[i]}_F E'$ for each i, it will suffice to prove that $A^i_G E' \subseteq A^{[i]}_F E'$ for each i. We work by induction on i. Assume that $e' \in A^i_G E'$. Then $\eta_G(e') \in M^i_G G^* E'$ and hence $\varepsilon(\eta_G(e')) \in M^i_F F^* E'$. Then

$$\eta_F(e') = \varepsilon(\eta_G(e')) - \sum_{j>0} p^{[j]} g^j \eta_F(\partial^j(e')).$$

By the Griffiths transversality of the filtration $A^\bullet_G$ (proved in Proposition 3), each $\partial^j(e')$ lies in $A^{i-j}_G E'$, hence the induction hypothesis implies that $\partial^j(e') \in A^{[i-j]}_F E'$ when $j > 0$. Then $p^{[j]}\partial^j(e') \in A^{[i]}_F E'$, and we conclude that $\eta_F(e') \in F^*(A^{[i]}_F E') + M^i_F F^* E' = M^{[i]}_F F^* E'$ by the second equation of Proposition 6. Its last equation then implies that $e' \in A^{[i]}_F E'$. ∎

EXAMPLE. Let $X := \operatorname{Spec} k[t, t^{-1}]$, let Z be the p-adic completion of $\operatorname{Spec} W[t, t^{-1}]$ and let F send t to t^p. Let E be the free $\mathcal{O}_Z$-module with basis $(e_0, \ldots, e_{p-1})$, let θ be the endomorphism of E sending e_i to e_{i-1} and e_0 to zero, and let

$$\nabla e_i = \theta(e_i) dt/t.$$

Recall that

$$\log(1 + x) = x - x^2/2 + x^3/3 + \cdots .$$

For each i, we have a formal horizontal section

$$\begin{aligned} \tilde{e}_i &:= e^{-\theta \log t}(e_i) \\ &:= e_i + \frac{(-\log t)e_{i-1}}{1!} + \frac{(-\log t)^2 e_{i-2}}{2!} + \cdots + \frac{(-\log t)^i}{i!} e_0. \end{aligned}$$

Then

$$\begin{aligned} e_i &= e^{\theta \log t}(\tilde{e}_i) \\ &= \tilde{e}_i + \frac{(\log t)\tilde{e}_{i-1}}{1!} + \frac{(\log t)^2 \tilde{e}_{i-2}}{2!} + \cdots + \frac{(\log t)^i \tilde{e}_0}{i!}. \end{aligned}$$

Let $(E', \nabla') = (E, \nabla)$ and define

$$\tilde{\Phi}_F \colon F^*(E', \nabla') \to (E, \nabla) : e'_i \mapsto p^i e_i.$$

It is immediate to check that this map is horizontal

Now suppose that G is another lift of F_X, sending t to $t^p + pg$. Let $u := (1 + pt^{-p}g)$, so that $t^p + pg = t^p u$. Note that $\log u \in pW\{t, t^{-1}\}$, say $\log u = p\delta$. That is,

$$\delta = t^{-p}g - (p/2)t^{-2p}g^2 + (p^2/3)t^{-3p}g^3 + \cdots .$$

To calculate $\varepsilon := \varepsilon_{G,F}$, we use the fact that it acts as the identity on horizontal sections. Since $G^*(t) = t^p u$, we have

$$\begin{aligned}\varepsilon(G^*(e_i)) &= \varepsilon(G^*(e^{(\log t)\theta}\tilde{e}_i) = e^{\log(t^p u)\theta}\varepsilon(G^*(\tilde{e}_i)) = e^{\log(t^p u)\theta}F^*(\tilde{e}_i)\\ &= e^{\log(t^p u)\theta}F^*(e^{-(\log t)\theta})(e_i) = e^{\log(t^p u)\theta}(e^{-(\log t^p)\theta})(e_i) = e^{(\log u)\theta}e_i\\ &= e_i + (\log u)e_{i-1} + (\log u)^2/(2!)e_{i-2} + \cdots + (\log u)^i/i!e_0\\ &= e_i + p\delta e_{i-1} + p^{[2]}\delta^2 e_{i-2} + \cdots + p^{[i]}\delta^i e_0.\end{aligned}$$

After tensoring with $\mathbf{Q}$, we can write

$$\begin{aligned}\Phi_G(e_i) &= \tilde{\Phi}_F(\varepsilon(G^*(e_i)))\\ &= p^i(e_i + \delta e_{i-1} + \delta^{[2]}e_{i-2} + \cdots + \delta^{[i]}e_0).\end{aligned}$$

Thus, $e_i \in A^i_G E'$ if $i < p$, but $\Phi_G(e_p) = p^p(\cdots) + (p^{p-1}/(p-1)!)\delta^p e_0$, which might not belong to $A^p_G E'$.

Take for example $p = 3$ and $g = t^3$. Then $u = 4$, so

$$\log u = \log(1+3) = 3 - 3^2/2 + 3^3/3 + \cdots \equiv 3 \pmod 9,$$

and hence $\delta \equiv 1 \pmod 3$. Then

$$\begin{aligned}\Phi_G(e_0) &= e_0,\\ \Phi_G(e_1) &= 3e_1 + 3\delta e_0,\\ \Phi_G(e_2) &= 9e_2 + 9\delta e_1 + 9\frac{\delta^2 e_0}{2},\\ \Phi_G(e_3) &= 27e_3 + 27\delta e_2 + 27\frac{\delta^2 e_1}{2} + 27\frac{\delta^3 e_0}{6}.\end{aligned}$$

Thus e'_3 does not belong to $A^3_G E'$, although it does belong to $A^3_F E'$.

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Arthur Ogus
Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720, USA
E-mail: ogus@math.berkeley.edu
https://math.berkeley.edu/~ogus/