The saturated de Rham-Witt complex for schemes with toroidal singularities

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Abstract

We show that the new construction [4] of the *saturated* de Rham-Witt complex gives good answers for schemes X with "ideally toroidal" singularities. In particular, the complexes $\mathcal{W}_1\Omega^{\cdot}_X$ have components which are coherent over $\mathcal{O}_X$, and they agree with certain complexes already familiar in the study of de Rham and crystalline cohomology of log schemes. We compare the Nygaard filtrations to the Hodge and conjugate filtrations for log schemes and show that the corresponding spectral sequences degenerate in the proper liftable case, in dimensions less than p.

Introduction

Crystalline cohomology [1],[3], conceived and developed by Grothendieck and Berthelot, provides a very satisfactory p-adic cohomology theory for smooth proper schemes X over a perfect field k. In particular, the crystalline cohomology of such a scheme consists of finitely generated modules over the Witt ring W

of k, and the corresponding integral structure carries important geometric information. This theory does not behave well for schemes with even the simplest singularities, but further work by Berthelot and others has developed a theory of "rigid cohomology [2]", which seem well behaved quite generally. However, these theories have coefficients in $\mathbf{Q} \otimes W(k)$ and thus provide no information about p-adic lattice structures or torsion.

One of the key applications of the integral version of crystalline cohomology has been to the study of the action of Frobenius, which has revealed subtle connections between zeta-functions and Hodge numbers [17]. An especially powerful tool for this study has been the "de Rham-Witt complex" [11], which is a canonical complex of abelian sheaves on the Zariski topology of a smooth scheme X/k and which calculates its crystalline cohomology. Recently, Bhatt, Lurie, and Mathew have found a new construction [4] of this complex, called the *saturated de Rham-Witt complex,* which is in some ways simpler and more general. It is the aim of this note to show how this new construction provides reasonable answers for a wide class of singular schemes, including schemes with "ideally toroidal" singularities. These results apply to fine saturated and smooth idealized log schemes X/k. It turns out that the saturated de Rham Witt complex of X/k is quasi-isomorphic to certain de Rham complexes on suitable PD thickenings of X, and in particular to Danilov's de Rham complex of a smooth lifting Y/W. Applications include a version of the theorems of Deligne-Illusie (on Hodge to de Rham degeneration) and Mazur (on the Katz conjectures) for such a log scheme.

Here is a summary of the contents of this paper. The first section contains a review of the main constructions of [4], as well as a few additional perspectives, generalizations, and results that we will need later. Generalizing two key ideas from [4], we introduce the notions of "quasi-saturated" and "quasi-Cartier type." Corollary 1.9 shows that a quasi-isomorphism between quasi-saturated Dieudonné complexes induces an isomorphism between their strict saturations, a result which will be used later to prove that the saturated de Rham-Witt complex calculates crystalline cohomology (Theorem 6.8). Proposition 1.13 gives another such criterion, which will be used in the following section devoted to de Rham and de Rham-Witt complexes of monoid algebras.

Sections 2 and 3 contain our main technical results about Dieudonné complexes of monoid algebras. A monoid algebra A over the Witt ring has a natural Frobenius lifting ϕ, and its saturated de Rham-Witt complex can be obtained by applying the saturation functor to the Dieudonné complex deduced from the de Rham complex $\Omega^{\cdot}_{A/W}$ together with the action of ϕ. The singularities of A make this complex difficult to handle, and the technical key to our paper is the fact that the saturation of $\Omega^{\cdot}_{A/W}$ is the same as the saturation of the complex $\underline{\Omega}^{\cdot}_{A/W}$ of "Danilov" (or "Zariski") differentials, which is much better behaved. In Section 3 we generalize slightly by considering algebras which are quotients of monoid algebras by ideals generated by monoidal ideals. This additional flexibility allows us to handle some reducible schemes: those obtained by gluing monoidal schemes along faces in a suitable way. Theorem 3.4 summarizes the conclusions we can draw about the saturated de Rham-Witt complex of such a k-algebra A, including an explicit description of $\mathcal{W}_1\Omega^{\cdot}_A$. We also show in Theorem 3.5 that the saturated de Rham-Witt complex of such an algebra satisfies simplicial descent with respect to the normalization mapping.

Section 4 applies the results of the previous two sections to a global setting, answering several questions asked by Illusie [12, §4.3]. Theorem 4.1 shows that, for a scheme X/k which looks étale locally like an ideally toric scheme, the components of $\mathcal{W}_n\Omega^{\cdot}_X$ are coherent over $W_n\mathcal{O}_X$. On the other hand, $\mathcal{W}_n\Omega^{\cdot}_X$ is is not always obtained as the pushforward of the classical construction on the regular locus of X; instead one must pushforward along a locally toric resolution of singularities. We have not addressed the possible comparison between the saturated de Rham-Witt complex and rigid cohomology, an important question raised in [12, §6].

Section 5 discusses the relation between the saturated de Rham-Witt complex and crystalline cohomology. In particular, it describes how to construct $\mathcal{W}\Omega^{\cdot}_X$ from the PD-envelope of X in a suitable embedding in a smooth Y/W endowed with a Frobenius lifting. This gives a very direct proof of the comparison between crystalline and de Rham-Witt cohomology in the smooth case. (We should note that [4, Theorem 10.1.2] gives a a very general existence and uniqueness result for such an isomorphism.)

In Section 6 we discuss log schemes. The scheme underlying a fine and saturated and smooth idealized log scheme X/k has ideally toroidal singularities, and the version of the de Rham-Witt complex we are discussing here does not see the log structure. However, the additional information provided by a log structure allows us to compare this complex to the de Rham cohomology of a (log) smooth lifting Y/W and to the singular cohomology of its generic fiber (see Corollary 6.9). The proof of this comparison requires Theorem 6.7, a crystalline Poincaré lemma for the complexes of Danilov differentials in mixed characteristic. Results in this section also include a second proof of the crystalline to de Rham-Witt comparison theorem (Theorem 6.8) and also Theorem 6.10, a version of the Deligne-Illusie theorem for the saturated de Rham-Witt complex, in a logarithmic context.

In the last section we discuss the Nygaard filtration. We begin with a general definition, based on the "abstract " construction of Mazur [17], which is easy to explicate for Dieudonné complexes which are of Cartier type or are saturated (see Proposition 7.4). Theorem 7.5 is a filtered version of the key quasi-isomorphism theorem [4, 2.7.3], which turns out to also be a generalization of Nygaard's key theorem [19, 1.5]. Finally, we show in Proposition 7.9 that formation of Nygaard filtrations commutes with passage to hypercohomology under certain conditions and that these are often satisfied for smooth log schemes over W. This paper also includes a technical appendix explaining the difference between Danilov differentials and $\mathcal{W}_1\Omega^{\cdot}_X$ in small characteristics.

This paper owes a huge debt to Luc Illusie, whose enormous generosity with time, ideas, conversations, and guidance helped shape and motivate this project, and whose immense work on the original construction of the de Rham-Witt complex remains a classic inspiration. I first learned about the saturated

de Rham-Witt complex directly from him, and his paper [12] remains an invaluable guide. I am also grateful to Bhatt, Lurie, and Mathew, the three authors of [4], who patiently listened to some of my inchoate ideas at the early stages of the research and who provided useful feedback. Thanks also go to the anonymous referee, who pointed out many egregious errors and misprints in an earlier version of this manuscript.

1 Dieudonné complexes and Dieudonné algebras

Let us briefly review the main definitions of [4] and gather the basic facts that we will need about them.

Definition 1.1. *A Dieudonné complex is a triple* $(M^\cdot, d, F)$, *where* $(M^\cdot, d)$ *is a cochain complex of abelian groups and* $F\colon M^\cdot \to M^\cdot$ *is an endomorphism of the underlying graded abelian group such that* $dF = pFd$. *A Dieudonné algebra is a Dieudonné complex* $(A^\cdot, d, F)$ *endowed with a structure of a commutative differential graded algebra such that* $F(a) \equiv a^p \pmod p$ *for every* $a \in A^0$ *and such that* $A^n = 0$ *if* $n < 0$.

If R is a ring endowed with an endomorphism σ, then a *Dieudonné complex (*resp. *algebra) over* R is a Dieudonné complex (resp. algebra) in which $(M^\cdot, d)$ is complex (resp. differential graded algebra) of R-modules and F is σ-linear. The category of Dieudonné complexes admits kernels and cokernels in the obvious way and in fact is an abelian category. One can also consider Dieudonné complexes of sheaves on a topological space or topos, and we shall often do so without comment.

If $(M^\cdot, d, F)$ is a Dieudonné complex, the endomorphism F is not a morphism of complexes, but we can adjust for this in several ways. For example, F induces a morphism of complexes:

$$F\colon (M^\cdot, pd) \to (M^\cdot, d) \tag{1.1}$$

and hence, after reduction modulo p, a morphism of graded abelian groups;

$$\gamma\colon M^\cdot/pM^\cdot \to H^\cdot(M^\cdot/pM^\cdot, d) \tag{1.2}$$

Alternatively, let $\Phi^i\colon M^i \to M^i$ denote p^iF. Then $\Phi^\cdot\colon (M^\cdot, d) \to (M^\cdot, d)$ is a morphism of complexes. If the terms of $M^\cdot$ are p-torsion free, let

$$\eta(M)^i := \{\omega \in p^iM^i : d\omega \in p^{i+1}M^{i+1}\} \subseteq M^i[1/p].$$

Then $\Phi^\cdot$ factors through a morphism of complexes

$$\alpha\colon (M^\cdot, d) \to (\eta(M^\cdot), d). \tag{1.3}$$

The following proposition-definition is the basis of the new approach to the de Rham-Witt complex proposed in [4]. Its proof is immediate.

Proposition 1.2. *Let* $(M^\cdot, d, F)$ *be a Dieudonné complex each of whose terms is* p*-torsion free. Then the following conditions are equivalent.*

1. *The endomorphism F is injective, and an element x of $M^{\cdot}$ lies in its image if and only if $dx \in pM^{\cdot}$.*

2. *The morphism α is an isomorphism.*

If these conditions are satisfied, $(M^{\cdot}, d, F)$ is said to be saturated. □

Formation of the complex $\eta(M^{\cdot})$ can be understood as a special case of Deligne's construction of the "filtration décalée" [8, 1.3.3]. Let P denote the p-adic filtration of $M[1/p]$:

$$P^k M^i := p^k M^i \text{ for } i \in \mathbf{Z}.$$

Then $\eta(M)^i = \tilde{P}^0 M^i$, where $\tilde{P}$ is the décalée of the filtration P. Let us recall for convenience the definition and essential points of this construction.

Proposition 1.3 ([8, 1.3.3])**.** *If $(M^{\cdot}, P)$ is a filtered complex, let*

$$\tilde{P}^k M^i := \{\omega \in P^{i+k} M^i : d\omega \in P^{i+k+1} M^{i+1}\}.$$

1. *The obvious map*

$$\tilde{P}^k M^i \to \operatorname{Ker}\left(\operatorname{Gr}_P^{i+k} M^i \xrightarrow{d} \operatorname{Gr}_P^{i+k} M^{i+1}\right)$$

induces a map π which fits in an exact sequence of complexes:

$$\begin{tikzcd} 0 \arrow[r] & (K^{\cdot}_{P,k}, d) \arrow[r] & (\operatorname{Gr}^k_{\tilde{P}} M^{\cdot}, \operatorname{Gr}^k d) \arrow[r, "\pi"] \arrow[d, "="'] & (H^{\cdot}(\operatorname{Gr}_P^{\cdot+k} M^{\cdot}), \beta^{\cdot}) \arrow[r] \arrow[d, "="] & 0 \\ & & (E_0^{k,\cdot-k}(M^{\cdot}, \tilde{P}), d_0) \arrow[r] & (E_1^{\cdot+k,-k}(M^{\cdot}, P), d_1), & \end{tikzcd}$$

Here $K^i_{P,k} := (P^{i+k+1}M^i + dP^{i+k}M^{i-1})/\tilde{P}^{k+1}M^i$,

$$\beta^j \colon H^j(\operatorname{Gr}_P^{j+k} M^{\cdot}) \to H^{j+1}(\operatorname{Gr}_P^{j+k+1} M^{\cdot})$$

is the Bockstein map, and $E_{\cdot}^{\cdot,\cdot}(M, F)$ denotes the spectral sequence of a filtered complex (M, F).

2. *The complex $(K^{\cdot}_{P,k}, d)$ is acyclic, so π is a quasi-isomorphism.*

3. *If a morphism $\theta \colon (M^{\cdot}, P) \to (N^{\cdot}, Q)$ of filtered complexes induces a quasi-isomorphism $\operatorname{Gr}_P(M^{\cdot}) \to \operatorname{Gr}_Q(N^{\cdot})$, then it induces a quasi-isomorphism $\operatorname{Gr}_{\tilde{P}}(M^{\cdot}) \to \operatorname{Gr}_{\tilde{Q}}(N^{\cdot})$.*

Proof. Statements (1) and (2) are just unraveling the definitions. For (3), which comes from [4, 2.4.5], observe that if $\operatorname{Gr}(\theta)$ is a quasi-isomorphism, then the induced map of complexes $H^{\cdot}(\operatorname{Gr}_P^{\cdot+k} M, \beta) \to H^{\cdot}(\operatorname{Gr}_Q^{\cdot+k} N, \beta)$ is an isomorphism, hence a quasi-isomorphism for every k, and because the map π in the diagram is a quasi-isomorphism, the map

$$(\operatorname{Gr}^k_{\tilde{P}} M^{\cdot}, d) \to (\operatorname{Gr}^i_{\tilde{Q}} N^{\cdot}, d)$$

is also a quasi-isomorphism. □

In our case, since M is p-torsion free, multiplication by p^i induces an isomorphism $(H^\cdot(\mathrm{Gr}^0_P M^\cdot, d_1) \longrightarrow (H^\cdot(\mathrm{Gr}^i_P M^\cdot, d_1)$. Furthermore, one checks immediately that $\tilde{P}^1 M^\cdot = p\tilde{P}^0 M^\cdot$. Thus we find a commutative diagram:

$$\begin{tikzcd}
 & & (M^\cdot/pM^\cdot, d) \arrow[d, "\overline{\alpha}"'] \arrow[dr, "\gamma"] & & \\
0 \arrow[r] & (K^\cdot_{P,0}, d) \arrow[r] & (\eta(M^\cdot)/p\eta(M^\cdot), d) \arrow[r, "\pi'"] & (H^\cdot(M^\cdot/pM^\cdot), d_1) \arrow[r] & 0
\end{tikzcd} \tag{1.4}$$

Here π' is the composition of the quasi-isomorphism π with the isomorphism induced by p^{-i}, and γ is induced by F. This is the map of equation 1.2, now promoted to a morphism of complexes.

Let us consider the following conditions.

Definition 1.4. *A p-torsion free Dieudonné complex is:*

1. *saturated if α is an isomorphism.*
2. *quasi-saturated if α is a quasi-isomorphism,*
3. *of Cartier type if γ is an isomorphism,*
4. *of quasi-Cartier type if γ is a quasi-isomorphism.*

Since π is always a quasi-isomorphism, we see that $(M^\cdot, d, F)$ is of quasi-Cartier type if and only if $\overline{\alpha}$ is a quasi-isomorphism. Since $M^\cdot$ and $\eta(M^\cdot)$ are p-torsion free, this is true if $(M^\cdot, d, F)$ is quasi-saturated. If $M^\cdot$ is p-adically separated and complete, the converse also holds. Thus a p-torsion free and p-adically separated and complete Dieudonné complex is quasi-saturated if and only if it is of quasi-Cartier type. If Y/W is a formally smooth formal scheme over the Witt ring of a perfect field, with a Frobenius lifting ϕ_Y, the associated Dieudonné complex $(\Omega^\cdot_{Y/W}, d, F)$ is of Cartier type.

Remark 1.5. Illusie and Mathew have pointed out that if a Dieudonné complex is saturated, of Cartier type, and p-adically separated, then F is an isomorphism and $d = 0$. Indeed, if $(M^\cdot, d, F)$ is saturated, then $\overline{\alpha}$ is an isomorphism, and if it is of Cartier type, then γ is an isomorphism. It follows from diagram (1.4) that π is an isomorphism and hence that $K^\cdot_{P,0} = 0$. Now if $x \in M^i$, then $p^{i+1}x \in P^{i+1}M^i \subseteq p\tilde{P}^0 M^i$, so $dx \in pM^{i+1}$, and, since M is saturated, x belongs to the image of F. Thus F is surjective. Since dF^n is divisible by p^n and M is p-adically separated, it follows that $d = 0$.

The inclusion functor from the category of saturated Dieudonné complexes to the category of all Dieudonné complexes has a left adjoint $M \mapsto \mathrm{Sat}(M)$, which can be described in several convenient ways. If M is a Dieudonné complex, let

$$M[F^{-1}] := \varinjlim(M, F),$$

i.e., the localization of M by the endomorphism F of the graded abelian group underlying M. The differential d does not extend to $M[F^{-1}]$, but it does extend to $M[F^{-1}, p^{-1}]$, with $dF^{-n}(\omega) := p^{-n}F^{-n}(\omega)$.

Proposition 1.6. *If M is a Dieudonné complex, let M_{tf} be the quotient of M by its p-torsion submodule. Then M_{tf} is a Dieudonné complex, and $\mathrm{Sat}(M) = \mathrm{Sat}(M_{\mathrm{tf}})$. When M is p-torsion free, $\mathrm{Sat}(M)$ can be described in the following ways:*

1. $\mathrm{Sat}(M) = \varinjlim \big(M \xrightarrow{\alpha} \eta(M) \xrightarrow{\alpha} \eta^2(M) \cdots\big)$.
2. $\mathrm{Sat}(M) = \{\omega \in M[F^{-1}] : dF^n\omega \in p^nM$ *for some (equivalently for all)* $n \gg 0\}$.
3. $\mathrm{Sat}(M) := \{\omega \in M[F^{-1}] : d\omega \in M[F^{-1}]\}$.
4. $\mathrm{Sat}(M)$ *is the largest graded subgroup of* $M[F^{-1}]$ *closed under* d.

Proof. The proofs of (1), (2), and (3) can be found in [4, 2.3.1, 2.3.3, 2.3.4]. For (4), it will suffice to show that any graded subgroup N of $M[F^{-1}]$ which is stable under d is contained in $\mathrm{Sat}(M)$ (as defined by condition (2)). Indeed, if N is such a complex and $x \in N$, then also $dx \in N$ and consequently there exists some $n > 0$ such that F^nx and F^ndx both belong to M. Then $dF^nx = p^nF^ndx \in p^nM$, and hence $x \in \mathrm{Sat}(M)$. □

If M is a saturated Dieudonné complex and if $x \in pM$, then $dx \in pM$, so there is a unique x' such that $Fx' = x$. Thus, there is a unique additive homomorphism $V\colon M \to M$ such that $FV = p$; moreover $VF = FV = p$ and $Vd = pdV$. One checks immediately that

$$Fil^rM := dV^rM + V^rM$$

is stable under d and that $VFil^rM \subseteq Fil^{r+1}M$ and $Fil^rM \subseteq Fil^{r-1}M$. Let $\mathcal{W}_rM := M/Fil^rM$ and $\mathcal{W}M := \varprojlim \mathcal{W}_rM$, which inherits a natural structure of a Dieudonné complex.

Here is a summary of the key results about saturated Dieudonné complexes.

Proposition 1.7 (Higher Cartier isomorphisms)**.** *Let $(M^\cdot, d, F)$ be a saturated Dieudonné complex.*

1. *For each r, the map F^r induces an isomorphism of complexes:*

$$\overline{F}^r\colon \big(\mathcal{W}_rM^\cdot, d\big) \to \big(H^\cdot(M^\cdot/p^rM^\cdot), \beta\big)\,,$$

where β is the Bockstein differential.

2. *For each r, the natural projection induces a quasi-isomorphism*

$$\pi_r\colon (M^\cdot/p^rM^\cdot, d) \to (\mathcal{W}_rM^\cdot, d).$$

3. *For each r, the composition of $\overline{F}^r$ and $H^\cdot(\pi_r)$ defines an isomorphism of graded abelian groups:*

$$\psi_r \colon \mathcal{W}_r M^\cdot \to H^\cdot(\mathcal{W}_r M^\cdot, d).$$

Proof. See [4, 2.7.2,2.7.3] and [12, 5.1.3] for the statements about the underlying abelian groups, as well as Theorem 7.5 for a refined version of statement (2). The compatibility of the map in (1) with the differentials is the commutativity the diagram

$$\begin{tikzcd}
M^i/p^r M^i \arrow[r, "\overline{F}^r"] \arrow[d, "d"'] & H^i(M^\cdot/p^r M^\cdot) \arrow[d, "\beta"] \\
M^{i+1}/p^r M^{i+1} \arrow[r, "\overline{F}^r"] & H^{i+1}(M^\cdot/p^r M^\cdot).
\end{tikzcd}$$

Let us explain the straightforward calculation (up to sign). This Bockstein differential is the boundary map of the long exact sequence coming from the short exact sequence

$$0 \longrightarrow M^\cdot/p^r M^\cdot \xrightarrow{[p^r]} M^\cdot/p^{2r} M^\cdot \longrightarrow M^\cdot/p^r M^\cdot \longrightarrow 0.$$

Thus, if $y \in M^i$ lifts the class $\overline{y}$ of an element of $H^i(M^\cdot/p^r M^\cdot)$, then dy is divisible by p^r and the Bockstein of $\overline{y}$ is given by the class of $p^{-r}dy$. If $x \in M^\cdot$ and $y = F^r x$, then $p^{-r}dy = p^{-r}dF^r x = F^r dx$, as required. $\square$

The following key result is proved in [4, 2.4.2] when $n = 1$ under the stronger hypothesis that $(M^\cdot, d, F)$ be of Cartier type. That proof is easily adopted to cover this more general statement.

Theorem 1.8. *If $(M^\cdot, d, F)$ is a Dieudonné complex of quasi-Cartier type, then for every $n > 0$, the natural maps*

$$(M^\cdot/p^n M^\cdot, d) \to (\mathrm{Sat}(M^\cdot, d)/p^n \,\mathrm{Sat}(M^\cdot, d) \to (\mathcal{W}_n \,\mathrm{Sat}(M^\cdot, d))$$

and the maps

$$\varprojlim(M^\cdot/p^n M^\cdot, d) \to \varprojlim(\mathrm{Sat}(M^\cdot)/p^n \,\mathrm{Sat}(M^\cdot), d) \to \varprojlim(\mathcal{W}_n \,\mathrm{Sat}(M^\cdot), d)$$

are quasi-isomorphisms. If $(M^\cdot, d, F)$ is of Cartier type, then in addition the natural map of complexes:

$$(M^\cdot/pM^\cdot, d) \xrightarrow{\sim} \mathcal{W}_1 \,\mathrm{Sat}(M^\cdot, d).$$

is an isomorphism.

Proof. As we observed after Definition 1.4, if $(M^{\cdot}, d, F)$ is of quasi-Cartier type, then the map $\overline{\alpha}$ in diagram (1.4) is a quasi-isomorphism. As explained in (3) of Proposition 1.3, it then follows that $\eta(\alpha)$ induces a quasi-isomorphism

$$\eta(M^{\cdot})/p\eta(M^{\cdot}) \to \eta^2 M^{\cdot}/p\eta^2(M^{\cdot}),$$

and so on for every η^k. Passing to the limit, we conclude that the map

$$(M^{\cdot}/pM^{\cdot}, d)) \to (\mathrm{Sat}(M^{\cdot})/p\,\mathrm{Sat}(M^{\cdot}), d)$$

is a quasi-isomorphism. One deduces the analogous statement with p^n in place of p using induction on n, since $M^{\cdot}$ and $\mathrm{Sat}(M^{\cdot})$ are p-torsion free. Statement (2) of Proposition 1.7 tells us that the map

$$(\mathrm{Sat}(M^{\cdot}, d)/p^n\,\mathrm{Sat}(M^{\cdot}, d) \to \mathcal{W}_n\,\mathrm{Sat}(M^{\cdot}, d)$$

are quasi-isomorphism. The same holds in the limit because the transition maps are surjective. This concludes the proof of the first statement of the theorem.

If $(M^{\cdot}, d, F)$ is of Cartier type, then it is also of quasi-Cartier type, so the first statement holds again. Furthermore, the natural map

$$(M^{\cdot}, d) \to \mathrm{Sat}(M^{\cdot}, d) \to \mathcal{W}_1(\mathrm{Sat}\, M^{\cdot}, d)$$

factors through $M^{\cdot}/pM^{\cdot}$ and so induces the map in the second statement. To see that that map is an isomorphism, consider the commutative diagram of graded groups:

$$\begin{tikzcd} M^{\cdot}/pM^{\cdot} \arrow[r] \arrow[d, "\gamma"'] & \mathcal{W}_1\,\mathrm{Sat}(M^{\cdot}) \arrow[d, "\psi_1"] \\ H^{\cdot}(M^{\cdot}/pM^{\cdot}) \arrow[r] & H^{\cdot}(\mathrm{Sat}(M^{\cdot})/p\,\mathrm{Sat}(M^{\cdot})) \end{tikzcd}$$

The left vertical arrow (induced by F) is an isomorphism because $(M^{\cdot}, d, F)$ is of Cartier type, the bottom horizontal arrow is an isomorphism because $(M^{\cdot}, d, F)$ is of quasi-Cartier type, and the right vertical arrow is an isomorphism by Proposition 1.7. We conclude that the top horizontal arrow is also an isomorphism, as desired. $\square$

Corollary 1.9. *Let $\theta\colon (M^{\cdot}, d, F) \to (M'^{\cdot}, d, F)$ be a morphism of p-torsion free Dieudonné complexes. Suppose that θ is a quasi-isomorphism and that either $M^{\cdot}$ or $M'^{\cdot}$ is quasi-saturated (resp. of quasi-Cartier type). Then both complexes are quasi-saturated (resp. of quasi-Carier type), and the maps*

$$\mathcal{W}Sat(M^{\cdot}, d, F) \to \mathcal{W}Sat(M'^{\cdot}, d, F) \text{ and } \mathcal{W}_nSat(M^{\cdot}, d, F) \to \mathcal{W}_nSat(M'^{\cdot}, d, F)$$

are isomorphisms for all $n \geq 0$.

Proof. We have a commutative diagram:

$$\begin{tikzcd}
(M^{\cdot}, d) \arrow[r, "\theta"] \arrow[d, "\alpha"'] & (M'^{\cdot}, d) \arrow[d, "\alpha'"] \\
\eta(M^{\cdot}, d) \arrow[r, "\eta(\theta)"] & \eta(M'^{\cdot}, d)
\end{tikzcd}$$

Since θ is a quasi-isomorphism, so is $\eta(\theta)$. If either complex is quasi-saturated, then α or α' is a quasi-isomorphism, and hence both are. Thus both complexes are quasi-saturated, hence of quasi-Cartier type.

We also have a commutative diagram:

$$\begin{tikzcd}
(M^{\cdot}/pM^{\cdot}, d) \arrow[r] \arrow[d, "\gamma"'] & (M'^{\cdot}/pM'^{\cdot}, d) \arrow[d, "\gamma'"] \\
(H^{\cdot}(M^{\cdot}/pM^{\cdot}), d_1) \arrow[r] & (H^{\cdot}(M'^{\cdot}/pM'^{\cdot} d), d_1)
\end{tikzcd}$$

Since θ is a quasi-isomorphism and the complexes $M^{\cdot}$ and $M'^{\cdot}$ are p-torsion free, the top arrow is a quasi-isomorphism, which implies that the bottom arrow is actually an isomorphism. If either complex is of quasi-Cartier type, then one of the two vertical arrows is a quasi-isomorphism, and it follows that both must be, *i.e.*, that both complexes are of quasi-Cartier type.

Under either hypothesis, both complexes are of quasi-Cartier type, so Theorem 1.8 implies that the vertical maps in the following commutative diagram are quasi-isomorphisms.

$$\begin{tikzcd}
(M^{\cdot}/p^n M^{\cdot}, d) \arrow[r] \arrow[d] & (M'^{\cdot}/p^n M^{\cdot}, d) \arrow[d] \\
(\mathcal{W}_n Sat M^{\cdot}, d) \arrow[r] & (\mathcal{W}_n Sat M'^{\cdot}, d)
\end{tikzcd}$$

Since θ is a quasi-isomorphism and $M^{\cdot}$ and $M'^{\cdot}$ are p-torsion free, the top horizontal arrow is also a quasi-isomorphism, and hence so is the bottom arrow. Then it follows from (3) of Proposition 1.7 that the bottom maps are actually isomorphisms. $\square$

Definition 1.10. *A Dieudonné complex $(M^{\cdot}, d, F)$ or algebra is strict if it is saturated and the natural map $M \to \mathcal{W}M^{\cdot} := \varprojlim \mathcal{W}_n M$ is an isomorphism.*

It is proved in [4, 2.7.6] that if $(M^{\cdot}, d, F)$ is saturated, then in fact $(\mathcal{W}M^{\cdot}, d, F)$ is strict. We should also point out that statement (3) of Proposition 1.7 implies that any quasi-isomorphism of strict Dieudonné complexes is in fact an isomorphism.

If $M^\cdot$ is a saturated Dieudonné complex, then $pM^\cdot \subseteq VM^\cdot \subseteq Fil^1 M^\cdot$, and hence $\mathcal{W}_1 M^\cdot$ is annihilated by p. If $A^\cdot$ is a saturated Dieudonné algebra, then $Fil^1 A^\cdot$ is an ideal of $A^\cdot$ and hence $\mathcal{W}_1 A^0$ is an $\mathbf{F}_p$-algebra. Moreover, if $\mathcal{M}^\cdot$ is a saturated Dieudonné complex or algebra, then the same is true of $\mathcal{W}\mathcal{M}^\cdot$, which is in fact strict [4, 2.7.6].

We can now state the new construction (and version) of the de Rham-Witt complex described in [4, 4.1.5].

Theorem 1.11 ([4]). *The functor $A^\cdot \mapsto \mathcal{W}_1 A^0$, from the category of strict Dieudonné algebras to the category of $\mathbf{F}_p$-algebras, admits a left adjoint*

$$R \mapsto \mathcal{W}\Omega^\cdot_R.$$

If R is an $\mathbf{F}_p$-algebra, $\mathcal{W}\Omega^\cdot_R$ is called the saturated or strict de Rham-Witt complex of R. □

There are several constructions of $\mathcal{W}\Omega^\cdot_R$ presented in [4]. Here we describe the one most useful for our present purposes. For another, more general construction, see Theorem 5.2.

Proposition 1.12 ([4, 4.2.3]). *Let R be an $\mathbf{F}_p$-algebra which is the reduction modulo p of a p-torsion free ring $\tilde{R}$ admitting an endomorphism ϕ lifting the absolute Frobenius endomorphism of R.*

1. *The de Rham complex $\Omega^\cdot_{\tilde{R}}$ and its p-adic completion $\hat{\Omega}^\cdot_{\tilde{R}}$ admit a canonical structure F of a Dieudonné algebra.*

2. *The map*
$$(\mathcal{W}\Omega^\cdot_R, d, F) \to \mathcal{W}\,\mathrm{Sat}(\Omega^\cdot_{\tilde{R}}, d, F)$$
adjoint to the map $R = \tilde{R}/p\tilde{R} \to \mathcal{W}_1\,\mathrm{Sat}(\Omega^\cdot_{\tilde{R}})^0$ is an isomorphism, and similarly for $\hat{\Omega}^\cdot_{\tilde{R}}$. [1]

We next explain a few technical results which will help us with the computation of the saturation of Dieudonné complexes in some cases.

Proposition 1.13. *Let $\theta\colon \overline{M}^\cdot \to \underline{M}^\cdot$ be a morphism of Dieudonné complexes on a noetherian topological space, and suppose that $\underline{M}^\cdot$ is p-torsion free. Then the following conditions are equivalent.*

1. *The action of F on the kernel and cokernel of θ is locally nilpotent.*

2. *The induced map $\overline{M}^\cdot[F^{-1}] \to \underline{M}^\cdot[F^{-1}]$ is an isomorphism.*

3. *The induced map $\mathrm{Sat}(\theta)\colon\ \mathrm{Sat}(\overline{M}^\cdot) \to \mathrm{Sat}(\underline{M}^\cdot)$ is an isomorphism.*

Proof. Note first that if F is an endomorphism of an abelian group or sheaf M, then the kernel of the map $M \to M[F^{-1}]$ consists of the elements (or sections) of M which are annihilated by some power of F. In particular, $M[F^{-1}]$ vanishes

[1] Actually only the statement for the completion appears in [4], but the proof is the same in both cases.

if and only if F is locally nilpotent on M. Now let $A^{\cdot}$ (resp. $B^{\cdot}$) be the kernel (resp. cokernel) of θ, so that we have an exact sequence:

$$0 \to A^{\cdot} \to \overline{M}^{\cdot} \to \underline{M}^{\cdot} \to B^{\cdot} \to 0.$$

Since we are working on a noetherian topological space, the localization functor is exact, so the sequence

$$0 \to A^{\cdot}[F^{-1}] \to \overline{M}^{\cdot}[F^{-1}] \to \underline{M}^{\cdot}[F^{-1}] \to B^{\cdot}[F^{-1}] \to 0$$

is again exact. Thus the localization of θ is an isomorphism if and only if F is locally nilpotent on $A^{\cdot}$ and on $B^{\cdot}$. This proves that (1) and (2) are equivalent.

Suppose that (1) and (2) are verified. Since $\underline{M}^{\cdot}$ is p-torsion free, it follows that the p-torsion of $\overline{M}^{\cdot}$ is contained in the kernel of θ, hence that F is locally nilpotent on this p-torsion, and hence that the map from $\overline{M}^{\cdot}$ to its p-torsion free quotient induces an isomorphism after localization by F. Since this map also induces an isomorphism on saturations, we may replace $\overline{M}^{\cdot}$ by this quotient, and thus we may and shall assume that $M^{\cdot}$ is p-torsion free. We now have a commutative diagram in which the horizontal arrows are isomorphisms:

$$\begin{tikzcd}
\overline{M}^{\cdot}[F^{-1},p^{-1}] \arrow[r] & \underline{M}^{\cdot}[F^{-1},p^{-1}] \\
\overline{M}^{\cdot}[F^{-1}] \arrow[r] \arrow[u] & \underline{M}^{\cdot}[F^{-1}] \arrow[u]
\end{tikzcd}$$

The objects in the top row are endowed with the operator d, and by the characterization of the saturation functor Sat in (3) of Proposition 1.6, we see that $\mathrm{Sat}(\theta)$ is an isomorphism also.

Finally, observe that for any p-torsion free M, the maps $M \to \mathrm{Sat}(M) \to M[F^{-1}]$ induce isomorphisms after localization by F, and so (3) implies (2). $\square$

Corollary 1.14. *Suppose we are given a morphism of short exact sequences of p-torsion free Dieudonné complexes on a noetherian topological space*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0,
\end{tikzcd}$$

and suppose that any two of the vertical arrows induce an isomorphism after saturation. Then the same is true of the third vertical arrow.

Proof. The two given sequences remain exact after localization by F. If any two of the vertical arrows become isomorphisms on saturation, then Proposition 1.13

implies that they remain isomorphisms after localization, and then it follows that the third is also an isomorphism after localization. Applying the proposition again, we see that that arrow also becomes an isomorphism after saturation. $\square$

Proposition 1.15. *If $0 \to A^\cdot \to B^\cdot \to C^\cdot$ is an exact sequence of p-torsion free Dieudonné complexes on a noetherian topological space, then $0 \to \mathrm{Sat}(A^\cdot) \to \mathrm{Sat}(B^\cdot) \to \mathrm{Sat}(C^\cdot)$ is also exact.*

Proof. Since $C^\cdot$ is p-torsion free, $p^nB^\cdot \cap A^\cdot = p^nA^\cdot$ for all n. Thus the p-adic filtration of $B^\cdot$ induces the p-adic filtration $A^\cdot$.

Suppose $a \in A^n$ and $da \in p^{n+1}B^{n+1}$. Then da maps to zero in C^{n+1} and hence $da \in A^{n+1} \cap p^{n+1}B^n = p^{n+1}A^{n+1}$. It follows that $\eta(B^\cdot) \cap A^\cdot = \eta(A^\cdot)$, and hence that the sequence $0 \to \eta(A^\cdot) \to \eta(B^\cdot) \to \eta(C^\cdot)$ is exact. The same is true with η^n in place of η, by induction. Taking the direct limit, we find the conclusion. $\square$

Remark 1.16. Let $\mathbf{DC}$ be the category of Dieudonné complexes and $\mathbf{DC}_{sat}$ the full subcategory of saturated ones. It follows from Proposition 1.15 that the kernel of a homomorphism θ of saturated Dieudonné complexes is again saturated, but this is not true for its cokernel. However, since Sat is left adjoint to the inclusion functor, it is true that $\mathrm{Sat}(\mathrm{Cok}(\theta))$ is the cokernel of θ in the category $\mathbf{DC}_{sat}$. Thus $\mathbf{DC}_{sat}$ admits kernels and cokernels, although it is not abelian. The composite functor $inc \circ \mathrm{Sat} : \mathbf{DC} \to \mathbf{DC}_{sat} \to \mathbf{DC}$ is left exact, but not right exact. For an example, consider the Dieudonné complex $B^\cdot$, whose component in degree 0 is freely generated by elements b_n in degree zero and by b'_n, a_n in degree 1, for $n \in \mathbf{N}$, where $F(x_n) = x_{n+1}$ for $x = a, b, b'$, and where $da_n = db'_n = 0$ and $db_n = p^{n+1}b'_n + p^na_n$. Then $\{a_n : n \in \mathbf{N}\}$ forms a sub-Dieudonné complex A of B, and the quotient C of B by the subcomplex A is freely generated in degree 0 by the images c_n of b_n and in degree 1 by the images c'_n of b'_n. The localization of these Dieudonné complexes by F is exhibited by writing the same formulas with $n \in \mathbf{Z}$. One sees then that $c_{-1} \in \mathrm{Sat}(C)$ but is not in the image of $\mathrm{Sat}(B)$. On the other hand, it is true that the functor $\mathrm{Sat} \otimes \mathbf{Q}$ preserves surjectivity. Indeed, if $B^\cdot \to C^\cdot$ is surjective, then so is its localization by F, and $\mathrm{Sat}(C^\cdot) \otimes \mathbf{Q} = C^\cdot[F^{-1}]$.

2 Dieudonné complexes of monoid algebras

Recall that a commutative monoid Q is said to be *fine* if it is finitely generated and integral and that Q is said to be *toric* if it fine and saturated and in addition the associated abelian group Q^{gp} is free. If Q is a commutative integral monoid and R is a commutative ring, we denote by $R[Q]$ the monoid algebra of Q over R. Thus $R[Q]$ is the free R-module with basis

$$\beta \colon Q \to R[Q] : q \mapsto e^q,$$

and β is a homomorphism from Q to the multiplicative monoid underlying $R[Q]$. Endow $R[Q]$ with the natural Q^{gp}-grading in which e^q has degree q. For each

$q \in Q$, the degree q component of $R[Q]$ is a free R-module of rank one, with basis e^q, and is zero if $q \in Q^{\mathrm{gp}} \setminus Q$. Then the de Rham complexes $(\Omega^{\cdot}_{R[Q]}, d)$ and $(\Omega^{\cdot}_{R[Q^{\mathrm{gp}}]}, d)$ inherit a natural Q^{gp}-grading for which the differential preserves degrees. Exterior multiplication on these de Rham complexes gives them the structure of a strictly commutative differential graded algebra. To avoid confusing the two different gradings, we will say "Q^{gp}-grading" for the grading induced by the Q^{gp}-grading of the ring $R[Q]$ when necessary.

We fix a prime number p and assume that R is endowed with an endomorphism σ such that $\sigma(r) \equiv r^p \pmod p$ for every $r \in R$. For example, we could take $R = \mathbf{Z}$ and $\sigma = \mathrm{id}_{\mathbf{Z}}$, or R could be the Witt ring of a perfect field k and σ its Frobenius endomorphism.

Proposition 2.1. *Let Q be an integral monoid and let R be a p-torsion free commutative ring endowed with an endomorphism σ as above. Then there is a unique σ-homomorphism of R-algebras:*

$$F\colon \Omega^{\cdot}_{R[Q]/R} \to \Omega^{\cdot}_{R[Q]/R}$$

with the following properties.

1. *For each $q \in Q$, we have $F(e^q) = e^{pq}$ and $F(de^q) = e^{(p-1)q} de^q$.*
2. *For each $f \in R[Q]$, we have $F(df) = f^{p-1}df + d\left(\frac{F(f)-f^p}{p}\right)$.*

In particular, in degree zero F is the homomorphism:

$$\phi\colon \sum r_q e^q \mapsto \sum \sigma(r_q) e^{pq}.$$

The triple $(\Omega^{\cdot}_{R[Q]/R}, d, F)$ is a Dieudonné algebra.

Proof. The map $Q \to Q$ sending q to pq is a monoid homomorphism. Thus there is a unique homomorphism of R-algebras $R[Q] \to R[Q]$ sending e^q to e^{pq} for every $q \in Q$. The composition of this homomorphism with the homomorphism $R[Q] \to R[Q]$ sending $\sum r_q e^q$ to $\sum \sigma(r_q) e^q$ is the homomorphism ϕ shown above; furthermore ϕ is the unique σ-homomorphism such that $F(e^q) = e^{pq}$ for all q. Moreover, $\phi(f)$ is congruent to f^p mod p for every $f \in R[Q]$. Since $R[Q]$ is p-torsion free, we can conclude from [4, 3.2.1] that there is a unique endomorphism F of the ring $\Omega^{\cdot}_{R[Q]/R}$ such that $F(f) = \phi(f)$ for all $f \in R[Q]$ and such that condition (2) above holds. If $f = e^q$ for some $q \in Q$, then $F(f) = f^p$, and so formula (2) reduces to the second formula in (1).

Formula (2) implies that $dF(f) = pFd(f)$ for every $f \in R[Q]$ and consequently that $dF = pFd$ on all of $\Omega^{\cdot}_{R[Q]/R}$. In fact, [4, 3.2.1] shows that $(\Omega^{\cdot}_{R[Q]}, d, F)$ is a Dieudonné algebra. $\square$

The complex $\Omega^{\cdot}_{R[Q]/R}$ seems hard to compute in general. We shall find it useful to consider some variants, which appear in various guises in the literature. First recall [20, V§2.2] that the map

$$R[Q] \to R[Q] \otimes Q^{\mathrm{gp}} \colon e^q \to e^q \otimes q$$

is a derivation and therefore induces a homomorphism of Q-graded $R[Q]$-modules

$$\Omega^1_{R[Q]/R} \to R[Q] \otimes Q^{\mathrm{gp}}.$$

Here the grading on the right is inherited from the grading on $R[Q]$; the elements of Q^{gp} are viewed in degree zero. The above map induces a homomorphism of differential graded algebras:

$$\Omega^{\cdot}_{R[Q]/R} \to R[Q] \otimes \Lambda^{\cdot} Q^{\mathrm{gp}}, \tag{2.1}$$

where the differential on the right is wedge product with q in degree q:

$$d(e^q \otimes \omega) := e^q \otimes q \wedge \omega \tag{2.2}$$

Define:

$$F\colon R[Q] \otimes \Lambda^i Q^{\mathrm{gp}} \to R[Q] \otimes \Lambda^i Q^{\mathrm{gp}}$$

by

$$\sum_q a_q e^q \otimes \omega \mapsto \sum_q \sigma(a_q) e^{pq} \otimes \omega. \tag{2.3}$$

Note that for $q \in Q$ and $\omega \in \Lambda^i Q^{\mathrm{gp}}$,

$$\begin{aligned} dF(e^q \otimes \omega) &= d(e^{pq} \otimes \omega) \\ &= e^{pq} \otimes (pq \wedge \omega) \\ &= pe^{pq} \otimes (q \wedge \omega) \\ &= pFd(e^q \otimes \omega). \end{aligned}$$

Thus d and F endow $R[Q] \otimes \Lambda^{\cdot} R[Q]$ with the structure of a Dieudonné complex. Since F is compatible with multiplication and induces the Frobenius endomorphism of $R/pR[Q]$, this Dieudonné complex is in fact a Dieudonné algebra.

The following proposition justifies the definition 2.3.

Proposition 2.2. *The homomorphism (2.1)*

$$\Omega^{\cdot}_{R[Q]/R} \to R[Q] \otimes \Lambda^{\cdot} Q^{\mathrm{gp}}$$

is a homomorphism of Dieudonné algebras and is an isomorphism if Q is an abelian group.

Proof. If $q \in Q$, statement (1) of Proposition 2.1 says that $F(de^q) = e^{(p-1)q} de^q$, which the homomorphism (2.1) takes to $e^{(p-1)q} e^q \otimes q = e^{pq} \otimes q$. On the other hand, (2.1) maps de^q to $e^q \otimes q$, which (2.3) takes to $e^{pq} \otimes q$. Since F on $\Omega^{\cdot}_{R[Q]/R}$ and on $R[Q] \otimes \Lambda^{\cdot} Q$ are algebra homomorphisms over σ, it follows that (2.1) is compatible with F, hence is a homomorphism of Dieudonné algebras. If Q is an abelian group, it is well known that this homomorphism is an isomorphism of differential graded algebras; see for example [20, IV,1.1.5]. $\square$

If Q is an integral monoid, the complex

$$R[Q] \otimes \Lambda^{\cdot} Q^{\mathrm{gp}} \subseteq R[Q^{\mathrm{gp}}] \otimes \Lambda^{\cdot} Q^{\mathrm{gp}} \cong \Omega^{\cdot}_{R[Q^{\mathrm{gp}}]/R}$$

corresponds to the so-called "logarithmic differentials" [20, IV,§2.2]. It has several variants, many of which have appeared in various forms in the literature [6], [7], [14], [20]. However, see the appendix for some subtle technicalities concerning these constructions.

Definition 2.3. *Let Q be an integral commutative monoid, let R be a ring, and let G be a face of Q. For $q \in Q$, let $\langle G, q\rangle$ denote the face of Q generated by G and q and let $\langle q\rangle := \langle Q^*, q\rangle$.*

1. $\Omega^i_{R[Q]/R}(log) := \sum_{q\in Q} \Omega^i_{R[Q^{\mathrm{gp}}]/R,q} = R[Q] \otimes \Lambda^i Q^{\mathrm{gp}}$.
2. $\underline{\Omega}^i_{R[Q]/R} \subseteq \Omega^i_{R[Q]/R}(log)$ *is the Q^{gp}-graded submodule whose component in degree qQ is $R \otimes \Lambda^i\langle q\rangle^{\mathrm{gp}}$ if $q \in Q$ and is zero if $q \notin Q$.*
3. $\underline{\Omega}^i_{R[Q]/R}(G) \subseteq \Omega^i_{R[Q]/R}(log)$ *is the Q^{gp}-graded submodule whose component in degree $q \in Q$ is $R \otimes \Lambda^i\langle G, q\rangle^{\mathrm{gp}}$.*

These submodules are all stable under d, F, and exterior multiplication, and so define sub-Dieudonné algebras of $(\Omega^{\cdot}_{R[Q^{\mathrm{gp}}]/R}, d, F)$. Note that $\underline{\Omega}^{\cdot}_{R[Q]/R}(Q^*) = \underline{\Omega}^{\cdot}_{R[Q]/R}$ and that $\underline{\Omega}^{\cdot}_{R[Q]/R}(Q) = \Omega^{\cdot}_{R[Q]/R}(log)$. Furthermore, formation of these complexes commutes with arbitrary base change $R \to R'$, as follows from the construction. In particular, they are defined over $\mathbf{Z}[Q]$. If Q is fine, this ring is noetherian and $\mathbf{Z}[Q] \otimes \Lambda^i Q^{\mathrm{gp}}$ is a noetherian $\mathbf{Z}[Q]$-module, and it follows that each of the modules defined above is finitely generated over $\mathbf{Z}[Q]$. Since their formation is compatible with base change, this conclusion holds for every R.

Proposition 2.4. *If Q is an integral monoid, there is a functorial commutative diagram of Dieudonné algebras:*

$$\begin{tikzcd}
(\Omega^{\cdot}_{R[Q]/R}, d, F) \arrow[r, "\theta"] \arrow[d] & (\underline{\Omega}^{\cdot}_{R[Q]/R}, d, F) \arrow[d] \\
(\Omega^{\cdot}_{R[Q^{\mathrm{gp}}]/R}, d, F) \arrow[r, "\sim"] & (\underline{\Omega}^{\cdot}_{R[Q^{\mathrm{gp}}]/R}, d, F)
\end{tikzcd}$$

Proof. The left vertical arrow comes from the functoriality of the construction in Proposition 2.1 and the natural map $Q \to Q^{\mathrm{gp}}$, the right vertical arrow exists from the definitions, and the bottom horizontal isomorphism comes from Proposition 2.2. To see the existence of θ, it suffices to check that each map $\Omega^i_{R[Q]/R} \to \underline{\Omega}^i_{R[Q^{\mathrm{gp}}]/R}$ factors through $\underline{\Omega}^i_{R[Q]/R}$, and it suffices to do this when $i = 1$. This is clear: for each $q \in Q$, the image of de^q in $\Omega^1_{R[Q^{\mathrm{gp}}]/R}$ is $e^q \otimes q$, which lies in $\underline{\Omega}^1_{R[Q]/R}$. $\square$

The following proposition reveals an important advantage of the complex $\underline{\Omega}^{\cdot}_{R[Q]/R}$. It is proved in [20, V, 2.3.17], but we repeat the proof here for the reader's convenience. It is modeled on Kato's proof in [14] of the log Cartier isomorphism, which implies that $(\Omega^{\cdot}_{R[Q]/R}(log), d, F)$ is of Cartier type. We note also that Blickle has also established a version of the Cartier isomorphism for "Danilov differentials" [5], but a technical subtlety prevents it from applying here (see the appendix). Proposition 3.3 gives a generalization to the idealized case.

Proposition 2.5. *Suppose that R/pR is perfect and Q is toric. Then the Dieudonné complex $(\underline{\Omega}^{\cdot}_{Q/R}, F, d)$ is of Cartier type.* □

Proof. We must prove that for every j, the homomorphism (1.2)

$$\gamma\colon \underline{\Omega}^{j}_{R[Q]/R} \otimes \mathbf{F}_p \to H^j(\underline{\Omega}^{\cdot}_{R[Q]/R} \otimes \mathbf{F}_p, d)$$

induced by F is an isomorphism. As we have seen, formation of these complexes commutes with base change, so it suffices to treat the case when R is itself a perfect ring of characteristic p. Since the endomorphism ϕ of R is an isomorphism, we are reduced by base change to the case in which $R = \mathbf{F}_p$, in which case $\sigma = \mathrm{id}$. Note that F maps the component of degree q to the component of degree pq. Using formula (2.3), we can identify the map γ as the map

$$\oplus_q \mathbf{F}_p \otimes \Lambda^j \langle q\rangle^{\mathrm{gp}} \to \oplus_{q'} H^j(\mathbf{F}_p \otimes \Lambda^{\cdot}\langle q'\rangle^{\mathrm{gp}}, d)$$

induced by the identity map on $\langle q\rangle = \langle pq\rangle$. Formula 2.2 identifies the differential

$$\begin{array}{ccc}
\underline{\Omega}^{j}_{\mathbf{F}_p[Q]/\mathbf{F}_p,q'} & \xrightarrow{d} & \underline{\Omega}^{j+1}_{\mathbf{F}_p[Q]/\mathbf{F}_p,q'} \\
\cong\downarrow & & \downarrow\cong \\
\mathbf{F}_p \otimes \Lambda^j\langle q'\rangle^{\mathrm{gp}} & \longrightarrow & \mathbf{F}_p \otimes \Lambda^{j+1}\langle q'\rangle^{\mathrm{gp}}
\end{array}$$

as wedge product by q', which vanishes if $q' = pq$, so γ is indeed an isomorphism for every q. It remains only to show that $H^j(\underline{\Omega}^{\cdot}_{\mathbf{F}_p[Q]/\mathbf{F}_p,q'})$ vanishes if $q' \notin pQ$. Since Q is saturated, in fact such a q' does not belong to pQ^{gp}, and hence its image x in $\mathbf{F}_p \otimes Q^{\mathrm{gp}}$ is not zero and belongs to a basis of the vector space $\mathbf{F}_p \otimes \langle q'\rangle^{\mathrm{gp}}$. As is well known, it follows that the complex

$$(\mathbf{F}_p \otimes \Lambda^{\cdot}\langle q'\rangle^{\mathrm{gp}}, \wedge x)$$

is acyclic. □

Remark 2.6. The saturation hypothesis on Q is not superfluous. For example, let Q be the monoid given by generators a, b with relation $a^n = b^n$. Then $Q^{\mathrm{gp}} \cong \mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z}$, and the homomorphism sending a to $(1,0)$ and b to $(1,\overline{1})$ identifies Q with $\{(x,t) \in \mathbf{Z} \oplus \mathbf{Z}/n\mathbf{Z} : x \geq \min(t \cap \mathbf{N})\}$. Then $\underline{\Omega}^{\cdot}_{Q/k}$ need not be of Cartier type even if k is of characteristic p relatively prime to n. The Cartier condition will fail unless whenever $q \in Q \setminus pQ$, the image of q in $Q^{\rm gp}/pQ^{\rm gp} \cong Q^{\rm gp}/(pQ^{\rm gp} + Q^{\rm gp}_{tors})$ is not zero. For example, if $n = 3$ and $p = 5$, let $q \in Q = (5, \overline{1})$. Then $q = 5(1, \overline{2}) = (5, \overline{1}) \in Q$, while $(1, \overline{2}) \in Q^{\rm gp} \setminus Q$.

The following theorem, which is our main computational tool, shows how the operation of saturation "cleans" the pathologies of the de Rham complex of a toroidal monoid. This result is enough to show that the saturated de Rham-Witt complex of schemes with toric singularities in the sense of [15] is well-behaved. We shall make this explicit in a more general context in Theorem 3.4 in the next section.

Theorem 2.7. *If Q is a toric monoid, the map*

$$\theta\colon (\Omega^{\cdot}_{R[Q]/R}, d, F) \to (\underline{\Omega}^{\cdot}_{R[Q]/R}, d, F)$$

induces isomorphisms:

$$\mathrm{Sat}(\Omega^{\cdot}_{R[Q]/R}, d, F) \to \mathrm{Sat}(\underline{\Omega}^{\cdot}_{R[Q]/R}, d, F)$$

and

$$\mathrm{Sat}(\hat{\Omega}^{\cdot}_{R[Q]/R}, d, F) \to \mathrm{Sat}(\hat{\underline{\Omega}}^{\cdot}_{R[Q]/R}, d, F),$$

where $\hat{M}$ means the p-adic completion of M.

Proof. It does not seem to be known whether or not the terms of the complex $\Omega^{\cdot}_{R[Q]/R}$ contain any p-torsion, but it is clear that $\underline{\Omega}^{\cdot}_{R[Q]/R}$ does not. In any case, by Proposition 1.13, it will suffice to show that the action of F on the kernel and cokernel of θ is nilpotent. To handle the cokernel, we will use the following explicit description of the image of θ.

Lemma 2.8. *Let $\overline{\Omega}^{\cdot}_{R[Q]/R}$ be the image of $\Omega^{\cdot}_{R[Q]/R}$ in $\underline{\Omega}^{\cdot}_{[Q]/R}$. Then with the $Q^{\rm gp}$-grading described in Definition 2.3, for each $q \in Q$, there is a natural identification*

$$\overline{\Omega}^{i}_{R[Q]/R,q} \cong R \otimes L_{i,q} \subseteq R \otimes \Lambda^i \langle q \rangle^{\rm gp} \cong \underline{\Omega}^{i}_{R[Q]/R,q},$$

where $L_{i,q}$ is the subgroup of $\Lambda^i Q^{\rm gp}$ generated by $\{q_1 \wedge \cdots \wedge q_i : q_1 + \cdots q_i \leq q\}$.

Proof. Recall that $\Omega^1_{R[Q]/R}$ is generated as an abelian group by elements of the form fdg for $f, g \in R[Q]$, and consequently $\Omega^i_{R[Q]/R}$ is generated by elements of the form $fdg_1 \wedge \cdots \wedge dg_i$. Writing f and each g_i as a sum $\sum a_q e^q$, we see that in fact $\Omega^i_{R[Q]/R}$ is generated as an R-module by elements of the form $\omega_{\mathbf{q}} := e^{q_0} de^{q_1} \wedge \cdots \wedge de^{q_i}$, where $\mathbf{q} := (q_0, q_1, \ldots, q_i)$. The degree of such an $\omega_{\mathbf{q}}$ is $q := q_0 + q_1 + \cdots q_i$, and so every element of $\Omega^i_{R[Q]/R,q}$ is a linear combination of such elements. The map $\Omega^1_{R[Q]/R} \to \underline{\Omega}^1_{R[Q]/R} \subseteq R[Q] \otimes Q^{\rm gp}$ takes de^q to $e^q \otimes q$ and hence $\omega_{\mathbf{q}}$ to $e^{q_0+q_1+\cdots+q_i} \otimes (q_1 \wedge \cdots \wedge q_i)$, which lies in $L_{i,q}$. This shows that $\overline{\Omega}^i_{R[Q]/R,q} \subseteq L_{i,q}$. On the other hand, if $q_1 + \cdots + q_i \leq q$, then $q_0 := q - (q_1 + \cdots + q_i) \in Q$, and we can let $\omega := e^{q_0} de^{q_1} \wedge \cdots \wedge de^{q_i}$, whose image in $\underline{\Omega}^i_{R[Q]/R,q}$ is $e^{q_0+q_1+\cdots q_i} \otimes q_1 \wedge \cdots \wedge q_i$. ∎

Proposition 2.9. *If Q is a toric monoid, let $(\overline{\Omega}^{\cdot}_{R[Q]/R}, d, F)$ and $(\underline{\Omega}^{\cdot}_{R[Q]/R}, d, F)$ be the associated Dieudonné complexes as described in Lemma 2.8 and Definition 2.3. Then the following statements hold.*

1. *For each $i \geq 0$ and each $q \in Q$, the map*

$$\overline{\Omega}^{i}_{R[Q],mq} \to \underline{\Omega}^{i}_{R[Q]/R,mq}$$

   is an isomorphism for m sufficiently large.

2. *There is an $n > 0$ such that*

$$F^n \underline{\Omega}^{i}_{R[Q]/R} \subseteq \overline{\Omega}^{i}_{R[Q]/R}.$$

3. *The action of F on the quotient $\underline{\Omega}^{\cdot}_{R[Q]/R}/\overline{\Omega}^{\cdot}_{R[Q]/R}$ is nilpotent.*

Proof. Note first that for $q \in Q$ and $m \in \mathbf{Z}^+$, $\langle q \rangle = \langle mq \rangle$. Lemma 2.8 identifies $\overline{\Omega}^{i}_{R[Q]/R}$ with $L_{i,mq} \subseteq \Lambda^i \langle q \rangle^{\mathrm{gp}}$. Every element of $\Lambda^i \langle q \rangle^{\mathrm{gp}}$ is a linear combination of elements of the form $q_1 \wedge \cdots \wedge q_i$ with each $q_i \in \langle q \rangle$, so each $q_i \leq m_i q$ for some $m_i \in \mathbf{N}$. But then $q_1 \wedge \cdots \wedge q_i \in L_{i,mq}$ for $m \geq m_1 + \cdots + m_i$. Thus $\Lambda^i \langle q \rangle^{\mathrm{gp}} = \cup \{L_{i,mq} : m \in \mathbf{N}\}$, and since $\Lambda^i \langle q \rangle^{\mathrm{gp}}$ is a noetherian abelian group, it follows that $L_{i,mq} = \Lambda^i \langle q \rangle^{\mathrm{gp}}$ for $m \gg 0$, proving statement (1). Suppose $\omega \in \underline{\Omega}^{i}_{R[Q]/R}$ has degree q. Then $F^n(\omega)$ has degree $p^n q$ and so statement (1) implies that it belongs to $\overline{\Omega}^{i}_{R[Q]/R}$ for $n \gg 0$. Any $\omega \in \underline{\Omega}^{i}_{R[Q]/R}$ can be written as a sum of homogenous elements, and it follows that F is locally nilpotent on $\underline{\Omega}^{i}_{R[Q]/R}/\overline{\Omega}^{i}_{R[Q]/R}$. As we have seen in the discussion after Definition 2.3, this $R[Q]$-module is finitely generated, and since F is ϕ-linear, its action is nilpotent, proving statements (2) and (3). $\square$

To finish the proof of Theorem 2.7, it will now suffice to prove the following proposition.

Proposition 2.10. *The kernel $A^{\cdot}_{R[Q]/R}$ of the natural map $\Omega^{\cdot}_{R[Q]/R} \to \underline{\Omega}^{\cdot}_{R[Q]/R}$ is stable under F, and the action of F on $A^{\cdot}_{R[Q]/R}$ is nilpotent.*

We shall need some preparatory lemmas. If Q is a monoid, we let Q^+ denote its maximal ideal, *i.e.*, $Q^+ = Q \setminus Q^*$.

Lemma 2.11. *If Q is sharp and $i \geq 1$, then $F\Omega^{i}_{R[Q]/R} \subseteq Q^+ \Omega^{i}_{R[Q]/R}$.*

Proof. Since $\Omega^{i}_{R[Q]/R} = \Lambda^i(\Omega^{1}_{R[Q]/R})$ and F is compatible with exterior multiplication, it suffices to prove this when $i = 1$. Since Q is sharp, $R[Q]$ is generated by $R[Q^+]$, and hence $\Omega^{1}_{R[Q]/R}$ is generated as an $R[Q]$-module by $\{de^q : q \in Q^+\}$. As we saw in Proposition 2.1,

$$F(de^q) = (e^q)^{p-1} de^q \in Q^+ \Omega^{1}_{R[Q]/R}. \quad \square$$

Lemma 2.12. *If Q is any toric monoid, then $FA^{\cdot}_{R[Q]/R} \subseteq Q^+ \Omega^{\cdot}_{R[Q]/R}$, and $F^n A^{\cdot}_{R[Q]/R} \subseteq Q^+ A^{\cdot}_{R[Q]/R}$ for n sufficiently large.*

Proof. Since Q is saturated, we can write $Q \cong \overline{Q} \oplus Q^*$, and we get a corresponding product decomposition $R[Q] \cong R[\overline{Q}] \otimes_R R[Q^*]$. If $q \in \overline{Q}, q^* \in Q^*$, and $q = \overline{q} + q^*$, then $\langle q\rangle^{\mathrm{gp}} = \langle \overline{q}\rangle^{\mathrm{gp}} \oplus Q^*$, so

$$
\begin{aligned}
&\underline{\Omega}^1_{R[Q]/R,q} = R \otimes \langle q\rangle^{\mathrm{gp}} \cong R \otimes \langle \overline{q}\rangle^{\mathrm{gp}} \oplus R \otimes \langle q^*\rangle^{\mathrm{gp}} \\
&\cong \underline{\Omega}^1_{R[\overline{Q}]/R,\overline{q}} \oplus \Omega^1_{R[Q^*]/R,q^*} \cong \Big(R[Q] \otimes_{R[\overline{Q}]} \underline{\Omega}^1_{R[\overline{Q}]/R} \oplus R[Q] \otimes_{R[Q^*]} \Omega^1_{R[Q^*]}\Big)_q.
\end{aligned}
$$

It follows that

$$
\underline{\Omega}^1_{R[Q]/R} \cong R[Q] \otimes_{R[\overline{Q}]} \underline{\Omega}^1_{R[\overline{Q}]/R} \oplus R[Q] \otimes_{R[Q^*]} \Omega^1_{R[Q^*]/R}
$$

Similarly,

$$
\Omega^1_{R[Q]/R} \cong R[Q] \otimes_{R[\overline{Q}]} \Omega^1_{R[\overline{Q}]/R} \oplus R[Q] \otimes_{R[Q^*]} \Omega^1_{R[Q^*]/R}
$$

$$
\Omega^i_{R[Q]/R} \cong \bigoplus_{a+b=i} (R[Q] \otimes_{R[\overline{Q}]} \Omega^a_{R[\overline{Q}]/R}) \otimes_R (R[Q] \otimes_{R[Q^*]} \otimes \Omega^b_{R[Q^*]/R})
$$

$$
\underline{\Omega}^i_{R[Q]/R} \cong \bigoplus_{a+b=i} (R[Q] \otimes_{R[\overline{Q}]} \underline{\Omega}^a_{R[\overline{Q}]/R}) \otimes_R (R[Q] \otimes_{R[Q^*]} \Omega^b_{R[Q^*]/R})
$$

Since $\Omega^b_{R[Q^*]/R} = \underline{\Omega}^b_{R[Q^*]/R}$ is free, since $R[\overline{Q}] \to R[Q]$ is flat, and since $A^0_{R[Q]/R} = 0$, we conclude that

$$
A^i_{R[Q]/R} \cong \bigoplus_{a+b=i, a\geq 1} R[Q] \otimes_{R[\overline{Q}]} A^a_{R[\overline{Q}]/R} \otimes_R R[Q] \otimes_{R[Q^*]} \Omega^b_{R[Q^*]/R}.
$$

The splitting $Q \to Q^*$, although not canonical, is compatible with F. It now follows from Lemma 2.11 that $FA^i_{R[Q]/R} \subseteq Q^+\Omega^i_{R[Q]/R}$. Then

$$
F^{n+1}A^i_{R[Q]/R} \subseteq F^n(Q^+\Omega^i_{R[Q]/R}) \subseteq (Q^+)^{p^n}\Omega^i_{R[Q]/R}.
$$

Since $A^i_{R[Q]/R}$ is stable under F, in fact

$$
F^{n+1}A^i_{R[Q]/R} \subseteq (Q^+)^{p^n}\Omega^i_{R[Q]/R} \cap A^i_{R[Q]/R}.
$$

If R is noetherian, the Artin-Rees lemma implies that there exists an n_0 such that $(Q^+)^{m+n_0}\Omega^i_{R[Q]/R} \cap A^i_{R[Q]/R} \subseteq (Q^+)^m A^i_{R[Q]/R}$ for all $m > 0$. In particular this holds when R is the localization $\mathbf{Z}_{(p)}$ of $\mathbf{Z}$ at p. Since our general R is by assumption flat over $\mathbf{Z}_{(p)}$, formation of $A^i_{R[Q]/R}$ and of these intersections commutes with base extension $\mathbf{Z}_{(p)} \to R$, and hence the same containment holds in general. We conclude that that $F^n A^{\cdot}_{R[Q]/R} \subseteq Q^+ A^{\cdot}_{R[Q]/R}$ for n sufficiently large. $\square$

If G is a face of Q, we let Q_G denote the localization of Q by G, which is toric if Q is. The next lemma shows that formation of the Dieudonné complexes we are considering is compatible with localization.

Lemma 2.13. *If G is a face of Q, the natural maps of Dieudonné complexes*

$$(\Omega^{\cdot}_{R[Q]/R})_G \to \Omega^{\cdot}_{R[Q_G]/R},$$

$$(\underline{\Omega}^{\cdot}_{R[Q]/R})_G \to \underline{\Omega}^{\cdot}_{R[Q_G]/R},$$

$$(A^{\cdot}_{R[Q]/R})_G \to A^{\cdot}_{R[Q_G]/R}$$

are isomorphisms.

Proof. It is clear that $R[Q_G] \cong R[Q]_G$, and since formation of de Rham complexes is compatible with localization, the first of the above maps is an isomorphism. In general, if M is any $Q^{\rm gp}$-graded $R[Q]$-module and G is a face of M, then M_G is again $Q^{\rm gp}$-graded, and if $x \in Q^{\rm gp}$, there is a natural identification $(M_G)_x \cong \varinjlim\{M_x \to M_{gx} : g \in G\}$ [20, I,3.2.8]. Moreover, for each $q \in Q$, $\langle q, G\rangle^{\rm gp}$ is the group envelope of the face of Q_G generated by q, and is also $\varinjlim\{\langle q\rangle^{\rm gp} \to \langle gq\rangle^{\rm gp} : g \in G\}$. Taking exterior powers, we deduce the second isomorphism, and the third then follows by a diagram chase. □

Proof of Proposition 2.10. We proceed by induction on the dimension of $\overline{Q}$. If this dimension is zero, then $Q^* = Q$, so $A^{\cdot}_{R[Q]/R} = 0$ and there is nothing to prove. If G is a nontrivial face of Q, then the dimension of Q/G is less than that of $\overline{Q}$, and so the induction hypothesis implies that the action of F on $A^{\cdot}_{R[Q_G]/R}$ is nilpotent: there is an n such that $F^n A^{\cdot}_{R[Q_G]/R} = 0$. Since Q has only finitely many faces, we can choose n independent of G. By Lemma 2.13, we conclude that

$$\left(F^n A^{\cdot}_{R[Q]/R}\right)_G = F^n\left((A^{\cdot}_{R[Q]/R})_G\right) = F^n\left(A^{\cdot}_{R[Q_G]/R}\right)$$

vanishes, for every nontrivial face of G.

Each nontrivial face G of Q defines an open subset Spec $R[Q]$, and the union of these open sets is the complement of the closed subset defined by the ideal Q^+. (This is just because every element of Q^+ generates a nontrivial face of Q.) Thus $F^n A^{\cdot}_{R[Q]/R}$ is supported at the ideal $R[Q^+]$: every element of $F^n A^{\cdot}_{R[Q]/R}$ is annihilated by some power of Q^+. Since this $R[Q]$-module is finitely generated, it is annihilated by $(Q^+)^{n'}$ for some $n' > 0$. On the other hand, by Lemma 2.12, there is an m such that $F^m A^{\cdot}_{R[Q]/R} \subseteq Q^+ A^{\cdot}_{R[Q]/R}$. Then for every $i > 0$,

$$F^{mi+n}\left(A^{\cdot}_{R[Q]/R}\right) \subseteq F^n\left((Q^+)^i A^{\cdot}_{R[Q]/R}\right) \subseteq (Q^+)^i F^n\left(A^{\cdot}_{R[Q]/R}\right),$$

which vanishes for i large enough. □

We have now proved that the kernel and cokernel of the map

$$\theta \colon (\Omega^{\cdot}_{R[Q]/R}) \to \underline{\Omega}^{\cdot}_{R[Q]/R}$$

are annihilated by some power of F. The same holds after p-adic completion. Thus Proposition 1.13 implies that $\mathrm{Sat}(\theta)$ is an isomorphism, and similarly for the p-adically completed complexes. □

3 Idealized monoid algebras

Theorem 2.7 can be extended to apply to some reducible toroidal schemes. Continuing with the hypotheses of Proposition 2.1, let K be an ideal of Q, let $R[K] \subseteq R[Q]$ denote the free R-module spanned by K, which forms an ideal of $R[Q]$, and let $R[Q,K]$ denote the quotient $R[Q]/R[K]$. Our aim is to compare the saturation of the de Rham complex of $\operatorname{Spec} R[Q,K]$ with a corresponding quotient of $\underline{\Omega}^{\cdot}_{R[Q]/R}$.

Let $\underline{\Omega}^{\cdot}_{R[K]/R} := \oplus\{\underline{\Omega}^{\cdot}_{R[Q]/R,k} : k \in K\}$, which forms a differential ideal in $\underline{\Omega}^{\cdot}_{R[Q]/R}$, and let $\underline{\Omega}^{\cdot}_{R[Q,K]/R}$ denote the quotient complex, which can be identified with $\oplus\{\underline{\Omega}^{\cdot}_{R[Q]/R,q} : q \in Q \setminus K\}$. Finally, let

$$\Omega^{\cdot}_{R[K]/R} := \operatorname{Ker}\left(\Omega^{\cdot}_{R[Q]/R} \to \Omega^{\cdot}_{R[Q,K]/R}\right),$$

a differential ideal in $\Omega^{\cdot}_{R[Q]/R}$.

Theorem 3.1. *The obvious maps fit into a diagram of short exact sequences of Dieudonné complexes:*

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{\cdot}_{R[K]/R} \arrow[r] \arrow[d, "\phi"'] & \Omega^{\cdot}_{R[Q]/R} \arrow[r] \arrow[d, "\theta"'] & \Omega^{\cdot}_{R[Q,K]/R} \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \underline{\Omega}^{\cdot}_{R[K]/R} \arrow[r] & \underline{\Omega}^{\cdot}_{R[Q]/R} \arrow[r] & \underline{\Omega}^{\cdot}_{R[Q,K]/R} \arrow[r] & 0
\end{tikzcd}$$

The action of F on the kernel and cokernel of each of the vertical arrows is nilpotent. Consequently, the maps ϕ, θ, and ψ induce isomorphisms

$$\begin{aligned}
\operatorname{Sat}(\Omega^{\cdot}_{R[K]/R}) &\xrightarrow{\sim} \operatorname{Sat}(\underline{\Omega}^{\cdot}_{R[K]/R}) \\
\operatorname{Sat}(\Omega^{\cdot}_{R[Q]/R}) &\xrightarrow{\sim} \operatorname{Sat}(\underline{\Omega}^{\cdot}_{R[Q]/R}) \\
\operatorname{Sat}(\Omega^{\cdot}_{R[Q,K]/R}) &\xrightarrow{\sim} \operatorname{Sat}(\underline{\Omega}^{\cdot}_{R[Q,K]/R}).
\end{aligned}$$

Proof. We have already treated the arrow θ. To establish the existence of the arrows ϕ and ψ, begin by observing that the kernel of the map

$$\Omega^1_{R[Q]/R} \to \Omega^1_{R[Q,K]}$$

is generated as an $R[Q]$-module by $K\Omega^1_{R[Q]/R}$ and dK. It follows that the differential ideal $\Omega^{\cdot}_{R[K]/R}$ is generated by $\{e^k : k \in K\}$. Then, since θ is a homomorphism of differential algebras and takes each e^k to an element of the differential ideal $\underline{\Omega}^{\cdot}_{R[K]/R}$, it induces the arrows ϕ and ψ.

We have already seen in Propositions 2.9 and 2.10 that F is nilpotent on the kernel and cokernel of θ, and it follows that it is also nilpotent on the kernel of ϕ and on the cokernel of ψ. The snake lemma yields an exact sequence:

$$\operatorname{Ker}(\theta) \to \operatorname{Ker}(\psi) \to \operatorname{Cok}(\phi) \to \operatorname{Cok}(\theta).$$

It will follow that F is nilpotent on the kernel of ψ and on the cokernel of ϕ provided we can prove that it is nilpotent on the cokernel of the map $\mathrm{Ker}(\theta) \to \mathrm{Ker}(\psi)$, which is isomorphic to the kernel of the map $\mathrm{Cok}(\phi) \to \mathrm{Cok}(\theta)$. The following lemma will allow us to conclude the argument.

Lemma 3.2. *Let $\overline{\Omega}^{\cdot}_{R[K]/R}$ denote the image of the map $\Omega^{\cdot}_{R[K]/R} \to \underline{\Omega}^{\cdot}_{R[K]/R}$. Then*

$$F\left(\overline{\Omega}^{\cdot}_{R[Q]/R} \cap \underline{\Omega}^{\cdot}_{R[K]/R}\right) \subseteq \overline{\Omega}^{\cdot}_{R[K]/R}.$$

Proof. Suppose that $\omega \in \overline{\Omega}^{i}_{R[Q]/R,k}$ with $k \in K$. Lemma 2.8 shows that ω is a linear combination of elements of the form $e^k \otimes q_1 \wedge \cdots \wedge q_i$, where $q_1 + \cdots + q_i \leq k$, and we may assume that ω itself has this form. Let $q_0 := k - (q_1 + \cdots + q_i)$. Then $k' := q_0 + (p-1)k \in K$, and so $e^{k'} de^{q_1} \wedge \cdots \wedge de^{q_i} \in \Omega^i_{R[K]/R}$. Then

$$\begin{aligned} F(\omega) &= e^{pk} \otimes q_1 \wedge \cdots \wedge q_i \\ &= e^{k' + q_1 + \cdots + q_i} \otimes q_1 \wedge \cdots \wedge q_i \\ &= \phi(e^{k'} de^{q_1} \wedge \cdots \wedge de^{q_i}), \end{aligned}$$

and so lies in $\overline{\Omega}^{i}_{R[K]/R}$. □

Now suppose that $\omega \in \underline{\Omega}^{i}_{R[K]/R}$ represents an element of the kernel of the map $\mathrm{Cok}(\phi) \to \mathrm{Cok}(\theta)$. Then its image in $\underline{\Omega}^{i}_{R[Q]/R}$ lies in $\overline{\Omega}^{i}_{R[Q]/R} \cap \underline{\Omega}^{\cdot}_{R[K]/R}$. By Lemma 3.2, $F(\omega)$ lies in the image of ϕ and hence vanishes in $\mathrm{Cok}(\phi)$. This concludes the proof that F is nilpotent on the kernel and cokernel of the vertical arrows in the diagram. The "consequence" then follows from Proposition 1.13. □

We next show that, if K is a radical ideal in Q, then the Dieudonné complex $(\underline{\Omega}^{\cdot}_{R[Q,K]/R}, d, F)$ is of Cartier type.

Proposition 3.3. *Suppose that R is p-torsion free and that R/pR is perfect. If K is a radical ideal in a toric monoid Q, the Dieudonné complexes $(\underline{\Omega}^{\cdot}_{R[Q]/R}, d, F)$, $(\underline{\Omega}^{\cdot}_{R[K]/R}, d, F)$, and $(\underline{\Omega}^{\cdot}_{R[Q,K]/R}, d, F)$ are of Cartier type.*

Proof. Recall that we have an exact sequence of Dieudonné complexes:

$$0 \to \underline{\Omega}^{\cdot}_{R[K]/R} \to \underline{\Omega}^{\cdot}_{R[Q]/R} \to \Omega^{\cdot}_{R[Q,K]/R} \to 0.$$

The complexes are Q^{gp}-graded, with the proviso that F multiplies degrees by p. Note that $\underline{\Omega}^{\cdot}_{R[K]/R}$ is just the submodule of $\underline{\Omega}^{\cdot}_{R[Q]/R}$ consisting of those terms whose degree q belongs to K. There is a natural splitting of this sequence obtained by identifying the quotient with the submodule of $\underline{\Omega}^{\cdot}_{R[Q]/R}$ whose degree q does not belong to K. This submodule is evidently closed under d, and it is also closed under F because K is a radical ideal. The map γ is then compatible with the direct sum decomposition, and since it is an isomorphism for $\underline{\Omega}^{\cdot}_{R[Q]/R}$, it must also be an isomorphism on each factor. □

Theorem 3.4. *Let K be a radical ideal in a toric monoid Q, let k be a perfect ring, and let R be a p-torsion free lift of k endowed with a lift σ of the Frobenius endomorphism of k.*

1. *The natural maps*

$$(\mathcal{W}\Omega^{\cdot}_{k[Q,K]}, d, F) \to \mathcal{W}\,\mathrm{Sat}(\Omega^{\cdot}_{R[Q,K]/R}, d, F) \to \mathcal{W}\,\mathrm{Sat}(\underline{\Omega}^{\cdot}_{R[Q,K]/R}, d, F)$$

and

$$(\mathcal{W}\Omega^{\cdot}_{k[Q,K]}, d, F) \to \mathcal{W}\,\mathrm{Sat}(\hat{\Omega}^{\cdot}_{R[Q,K]/R}, d, F) \to \mathcal{W}\,\mathrm{Sat}(\hat{\underline{\Omega}}^{\cdot}_{R[Q,K]/R}, d, F)$$

are isomorphisms.

2. *The natural map*

$$(\hat{\underline{\Omega}}^{\cdot}_{R[Q,K]}, d) \longrightarrow (\mathcal{W}\,\mathrm{Sat}(\underline{\Omega}^{\cdot}_{R[Q,K]}), d) \cong (\mathcal{W}\Omega^{\cdot}_{k[Q,K]}, d)$$

is a quasi-isomorphism.

3. *The natural map $\Omega^{\cdot}_{k[Q,K]/k} \longrightarrow \mathcal{W}_1\Omega^{\cdot}_{k[Q,K]/k}$ factors through an isomorphism*

$$\left(\underline{\Omega}^{\cdot}_{k[Q,K]/k}, d\right) \longrightarrow \left(\mathcal{W}_1\Omega^{\cdot}_{k[Q,K]/k}, d\right).$$

4. *For every $n \in \mathbf{N}$, let $R_n := R/p^nR$. Then there are isomorphisms of complexes:*

$$\begin{array}{rcl} \left(\mathcal{W}_n\Omega^{\cdot}_{k[Q,K]}, d\right) & \longrightarrow & \left(H^{\cdot}(\mathrm{Sat}\,\underline{\Omega}^{\cdot}_{R_n[Q,K]/R_n}, d), \beta\right) \\ \left(H^{\cdot}(\underline{\Omega}^{\cdot}_{R_n[Q,]/R_n}, d), \beta\right) & \longrightarrow & \left(H^{\cdot}(\mathrm{Sat}\,\underline{\Omega}^{\cdot}_{R_n[Q,K]/R_n}, d), \beta\right) \end{array}$$

and a quasi-isomorphism

$$\left(\underline{\Omega}^{\cdot}_{R_n[Q,K]/R_n}, d\right) \longrightarrow \left(\mathcal{W}_n\Omega^{\cdot}_{k[Q,K]/k}, d\right).$$

Proof. The ring $R[Q,K]$ is a flat lift of $k[Q]$ and ϕ lifts the absolute Frobenius endomorphism of $k[Q,K]$, so statement (2) of Proposition 1.12 asserts that the natural map $\mathcal{W}\Omega^{\cdot}_{k[Q,K]} \to \mathcal{W}\,\mathrm{Sat}(\Omega^{\cdot}_{R[Q,K]/R})$ is an isomorphism of Dieudonné algebras. Theorem 3.1 tells us that $\mathcal{W}\,\mathrm{Sat}(\Omega^{\cdot}_{R[Q,K]/R}) \to \mathcal{W}\,\mathrm{Sat}(\underline{\Omega}^{\cdot}_{R[Q,K]/R})$ is also an isomorphism, and similarly for the p-adic completions. This proves statement (1).

By Proposition 3.3, $(\underline{\Omega}^{\cdot}_{R[Q,K]/R}, d, F)$ is of Cartier type, so Theorem 1.8 implies statements (2) and (3). Since $(\underline{\Omega}^{\cdot}_{R[Q,K]/R}, d, F)$ is of quasi-Cartier type, the maps

$$(\underline{\Omega}^{\cdot}_{R_n[Q,K]/R_n}, d) \to \mathrm{Sat}(\underline{\Omega}^{\cdot}_{R_n[Q,K]/R_n}, d)$$

are quasi-isomorphisms, and so statement (4) follows from Proposition 1.7. $\square$

The algebras $R[Q,K]$ and the corresponding complexes $\underline{\Omega}^{\cdot}_{R[Q,K]/R}$ and $\mathcal{W}\Omega^{\cdot}_{k[Q,K]}$ have an appealing geometric interpretation. Recall first that if K is a prime ideal, its complement G is a face of Q, and the natural map $R[G] \to R[Q,K]$ is an isomorphism. In fact, it follows immediately from the definitions that

$$\underline{\Omega}^i_{R[G]} = \sum_{g \in G} \Lambda^i \langle g \rangle^{\mathrm{gp}} = \underline{\Omega}^i_{R[Q,K]},$$

so the map $\underline{\Omega}^{\cdot}_{R[G]} \to \underline{\Omega}^{\cdot}_{R[Q,K]}$ is also an isomorphism.

More generally, every radical ideal K is the intersection of a finite number of primes, and $\operatorname{Spec} R[Q,K]$ is a union of the spectra of the monoid algebras of the corresponding faces. We shall see that $\underline{\Omega}^{\cdot}_{R[Q,K]/R}$ and $\mathcal{W}\Omega^{\cdot}_{k[Q,K]}$ satisfy descent with respect to the gluing of these faces. First note that if K_1 and K_2 are prime ideals of Q, then so is their union K_{12}, and $R[K_{12}] = R[K_1] + R[K_2] \subseteq R[Q]$. Thus if $G_i := Q \setminus K_i$ and $G_{12} = G_1 \cap G_2$, we have an exact sequence:

$$0 \to R[Q,K] \to R[G_1] \oplus R[G_2] \to R[G_{12}] \to 0.$$

Indeed, if $q \in Q$, the degree q part of the sequence looks like:

$$0 \to 0 \to 0 \to 0 \to 0 \qquad \text{if } q \in K$$

$$0 \to \mathbf{Z} \to \mathbf{Z} \to 0 \to 0 \qquad \text{if } q \in G_i \setminus G_{12}$$

$$0 \to \mathbf{Z} \to \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z} \to 0 \qquad \text{if } q \in G_{12}\ .$$

In any of these cases, the sequence remains exact when tensored with $\Lambda^i \langle q \rangle^{\mathrm{gp}}$, and consequently the sequence

$$0 \to \underline{\Omega}^i_{R[Q,K]} \to \underline{\Omega}^i_{R[G_1]} \oplus \underline{\Omega}^i_{R[G_2]} \to \underline{\Omega}^i_{R[G_{12}]} \to 0$$

is also exact.

More generally, suppose that $K = K_1 \cap \cdots \cap K_m$, where each K_i is a prime ideal with complementary face G_i. The normalization of $R[Q,K]$ is the homomorphism

$$R[Q,K] \to \tilde{R}[Q,K] := \oplus_i R[Q,K_i] \cong \oplus_i R[G_i]$$

For any multi-index $I := (I_0, \dots I_n)$, let $K_I := K_{I_0} \cup \cdots \cup K_{I_n}$ and $G_I := Q \setminus K_I := G_{I_0} \cap \cdots G_{I_n}$. Consider the cosimplicial ring:

$$R_\bullet[Q,K] := R_0[Q,K] \rightrightarrows R_1[Q,K] \overset{\to}{\rightrightarrows} \cdots \tag{3.1}$$

whose nth term is the $n+1$-fold product

$$\begin{aligned} R_n[Q,K] \quad &:= \quad \tilde{R}[Q,K] \otimes_{R[Q,K]} \tilde{R}[Q,K] \cdots \otimes_{R[Q,K]} \tilde{R}[Q,K] \\ &\cong \quad \bigoplus_{|I|=n+1} R[Q,K_I] \cong \bigoplus_{|I|=n+1} R[G_I]. \end{aligned}$$

The face and degeneracy maps are the obvious ones, and in particular the map $R[Q,K] \to \oplus_i R[G_i]$ sends e^q to $\oplus_i \{e^q \in R[G_i] : q \in G_i\}$.

The following theorem shows that the saturated de Rham-Witt complex behaves as expected idealized monoid algebras and is a generalization of a result of Illusie [12, §4.1].

Theorem 3.5. *If K is a radical ideal of a toric monoid Q and $i \in \mathbf{N}$, let $C^\bullet(\underline{\Omega}^i_{R[Q,K]}, \partial)$ denote the cochain complex associated to the cosimplicial ring $R_\bullet[Q,K]$ (3.1). Then the augmentation map*

$$\underline{\Omega}^i_{R[Q,K]/R} \to C^\bullet(\underline{\Omega}^i_{R[Q,K]/R}, \partial)$$

is a quasi-isomorphism. Similarly, the maps

$$\mathcal{W}_n\Omega^i_{k[Q,K]} \to C^\bullet(\mathcal{W}_n\Omega^i_{k[Q,K]}, \partial)$$

and

$$\mathcal{W}\Omega^i_{k[Q,K]} \to C^\bullet(\mathcal{W}\Omega^i_{k[Q,K]}, \partial)$$

are quasi-isomorphisms.

Proof. Let $C^{-1}(\underline{\Omega}^i_{R[Q,K]/R}) := \underline{\Omega}^i_{R[Q,K]/R}$ and let $\partial^{-1} \colon \underline{\Omega}^i_{R[Q,K]/R} \to C^0(\underline{\Omega}^i_{R[Q,K]/R})$ be the augmentation map. Our claim is that the augmented complex $\tilde{C}^\bullet(\underline{\Omega}^i_{R[Q,K})$ obtained by inserting the terms in degree -1 is acyclic. This complex is again Q-graded, where

$$\begin{aligned}\tilde{C}^n(\underline{\Omega}^i_{R[Q,K]/R,q}) &= \bigoplus_{|I|=n+1} \underline{\Omega}^i_{R[G_I]/R,q} \\ &= \bigoplus\{\Lambda^i\langle q\rangle^{\mathrm{gp}} : |I| = n+1 : q \in G_I\},\end{aligned}$$

if $n \geq 0$, and

$$\tilde{C}^{-1}(\underline{\Omega}^i_{R[Q,K],q}) = \Lambda^i\langle q\rangle^{\mathrm{gp}}$$

With the convention that $G_\emptyset = Q$, this also holds for $n = -1$. For $\omega \in \tilde{C}^n(\underline{\Omega}^i_{R[G_I],q})$, the differential of this complex is given by:

$$(\partial\omega)_I := \sum_{k=0}^{n+1} (-1)^k \omega_{\epsilon_k(I)},$$

where ϵ_k is the kth face map.

To show that the complex is acyclic, we construct homotopy operators as follows. For each face F of Q which does not meet K, there is some $i_F \in [1,m]$ such that $F \subseteq G_{i_F}$. If $q \in Q \setminus K$, then $\langle q\rangle \cap K = \emptyset$, and we let $i_q := i_{\langle q\rangle}$. Thus $q \in G_{i_q}$, and $i_q = i_{nq}$ for every $n > 0$. If $I := (I_0, \ldots, I_n)$, let $s_q(I) := (i_q, I_0, \ldots, I_n)$. Now if $n \in \mathbf{N}$, define

$$\rho_{n,q} \colon \tilde{C}^n(\underline{\Omega}^i_{R[Q,K]})_q \to \tilde{C}^{n-1}(\underline{\Omega}^i_{R[Q,K]})_q : (\rho_{n,q}(\omega.))_I := \omega_{s_q(I)}.$$

This makes sense because $\omega_{s_q(I)} \in \Lambda^i\langle q\rangle^{\mathrm{gp}}$ and $q \in G_{s_q(I)} = G_{i_q} \cap G_I \subseteq G_I$. The following lemma shows that ρ is a homotopy operator with respect to the boundary operator ∂ and that it is compatible with the Dieudonné module structure of $\underline{\Omega}^{\cdot}_{R[Q,K]/R}$ and of each $\underline{\Omega}^{\cdot}_{R[G_I]/R}$.

Lemma 3.6. *The following identities hold:*

1. $\partial\rho + \rho\partial = \mathrm{id} : \tilde{C}^n(\underline{\Omega}^i_{R[Q,K]}) \to \tilde{C}^n(\underline{\Omega}^i_{R[Q,K]})$;
2. $d\partial = \partial d : \tilde{C}^n(\underline{\Omega}^i_{R[Q,K]}) \to \tilde{C}^{n+1}(\underline{\Omega}^{i+1}_{R[Q,K]})$;
3. $d\rho = \rho d : \tilde{C}^n(\underline{\Omega}^i_{R[Q,K]}) \to \tilde{C}^{n-1}(\underline{\Omega}^{i+1}_{R[Q,K]})$;
4. $F\rho = \rho F : \tilde{C}^n(\underline{\Omega}^i_{R[Q,K]}) \to \tilde{C}^{n-1}(\underline{\Omega}^i_{R[Q,K]})$.

Proof. The proof of (1) is a straightforward and standard calculation we shall not repeat, and (2) is a consequence of the naturality of the constructions. For (3), observe that, for every I, we have:

$$(\rho_{n,q}(d\omega.))_I = (\rho_{n,q}(q \wedge \omega.))_I = q \wedge \omega_{s_q(I)} = d\omega_{s_q(I)} = (d\rho_{n,q}(\omega))_I.$$

Statement (4) holds because $i_q = i_{pq}$ for every $q \in Q \setminus K$. □

Statement (1) of Lemma 3.6 implies the acyclicity of the augmented complex $\tilde{C}^\bullet(\underline{\Omega}^i_{R[Q,K]/R})$ and hence also statement (1) of Theorem 3.5. Statements (3) and (4) of the lemma imply that each ρ_n is fact a morphism of Dieudonné complexes and therefore extends to define homotopy operators

$$\tilde{C}^n \operatorname{Sat}(\underline{\Omega}^{\cdot}_{R[Q,K]}) \to \tilde{C}^{n-1} \operatorname{Sat}(\underline{\Omega}^{\cdot}_{R[Q,K]})$$

for all n. These homotopy operators necessarily commute with d, F, and V, and hence pass to the quotient complexes $\mathcal{W}_n\underline{\Omega}^{\cdot}_{R[Q,K]/R}$. It follows that the maps

$$\mathcal{W}_n\Omega^i_{k[Q,K]} \to C^\bullet(\mathcal{W}_n\Omega^i_{k[Q,K]}, \partial)$$

are quasi-isomorphisms. Since the transition maps in these inverse systems are surjective, the same is true after taking the inverse limit. □

4 Ideally toroidal schemes

Let k be a perfect field and let X/k be a k-scheme locally of finite type. We shall say that X/k is *toroidal* if étale locally on X, there exist a toric monoid Q and an étale map $X \to \operatorname{Spec}(k[Q])$. Note that such a scheme is necessarily normal. More generally, we shall say that X/k is *ideally toroidal* if, étale locally on X, there exist a toric monoid Q, an ideal K in Q, and an étale map $X \to \operatorname{Spec} k[Q,K]$. Our aim is to explain that the strict de Rham-Witt complex of such a scheme is well-behaved in various senses.

For simplicity, we assume henceforth that our schemes are reduced; recall however that, in general, the saturated de Rham Witt complexes associated to a scheme and its reduced subscheme are the same. (This fact is a consequence of [4, 6.5.2] and also of the easier [4, 3.6.1].)

Theorem 4.1. *Let X/k be a reduced ideally toroidal scheme, locally of finite type over a perfect field k of characteristic $p > 0$.*

1. *The natural map $\mathcal{O}_X \to \mathcal{W}_1\Omega^0_X$ is an isomorphism, and the sheaf $\mathcal{W}_1\Omega^i_X$ is a coherent sheaf of $\mathcal{O}_X$-modules. It is torsion free if X is in fact toroidal.*

2. *More generally, for $n > 0$, there is a natural isomorphism $W_n\mathcal{O}_X \to \mathcal{W}_n\Omega^0_X$, and the sheaf $\mathcal{W}_n\Omega^j_X$ is a coherent sheaf of $W_n\mathcal{O}_X$-modules.*

3. *Let $X' \to X$ be the normalization mapping and let $C^\bullet(\mathcal{W}_n\Omega^i_X)$ denote the cochain complex associated to the simplicial scheme associated to the morphism $X' \to X$ and the functor $\mathcal{W}_n\Omega^i$. The natural map*

$$\mathcal{W}_n\Omega^i_X \to C^\bullet(\mathcal{W}_n\Omega^i_X)$$

is a quasi-isomorphism. The analogous result also holds with $\mathcal{W}$ in place of $\mathcal{W}_n$.

Proof. Since the sheaves $\mathcal{W}_n\Omega^{\cdot}_X$ are compatible with étale localization [4, 5.3.5], these statements can be verified étale locally on X, and we may assume that $X = \operatorname{Spec} k[Q, K]$, where Q is a toric monoid and K is an ideal in Q. Then statement (1) follows from (3) of Theorem 3.4. (The $k[Q]$-modules in the complex $\underline{\Omega}^{\cdot}_{k[Q]}$ are evidently torsion free.)

Lemma 4.2. *Let K be a radical ideal in a toric monoid Q and let k be a field. Then $k[Q, K]$ is seminormal.*

Proof. Let $A := k[Q, K]$. According to Swan's characterization of semi-normality, it is enough to prove that if (x, y) is a pair of elements of A satisfying $x^2 = y^3$, then there exists a $t \in A$ such that $x = t^3$ and $y = t^2$. Since K is reduced, its complement is a union of faces $G_1, \ldots, G_n$ of Q, and the normalization A' of A can be identified with the direct product of the monoid algebras $A_i := k[G_i]$. Each of these is a normal integral domain, and hence for each i there is a t_i such that $x_i = t_i^3$ and $y_i = t_i^2$, where (x_i, y_i) is the image of (x, y) in A_i. The tensor product $A' \otimes_A A'$ can be identified with the direct product of the rings $A_{i,j} := A_i \otimes_A A_j \cong k[G_i \cap G_j]$, each of which is also an integral domain. It follows easily that the images of t_i and t_j in $A_{i,j}$ agree, hence, by the descent property proved in 3.5, that the element $t.$ of A' descends to A. □

It follows from Lemma 4.2 and [4, 3.6.2 and 6.5.2] that the map $W_n\mathcal{O}_X \to \mathcal{W}_n\Omega^0_X$ is an isomorphism. Statement (2) then follows; see [12, 8.8(c)]. Statement (3) follows from Theorem 3.5. □

Proposition 4.3. *If k is perfect and X/k is a proper k-scheme with ideally toroidal singularities, then the hypercohomology groups $H^i(X, \mathcal{W}\Omega^{\cdot}_X)$ and $H^i(X, \mathcal{W}_n\Omega^{\cdot}_X)$ for each n are finitely generated W-modules.*

Proof. Recall from Proposition 1.7 that, for every n, the natural map

$$(\mathcal{W}\Omega^{\cdot}_X / p^n \mathcal{W}\Omega^{\cdot}_X, d) \to (\mathcal{W}_n\Omega^{\cdot}, d)$$

is a quasi-isomorphism. Theorem 4.1 shows that the terms of the complex $\mathcal{W}_1\Omega^{\cdot}_X$ are coherent sheaves of $\mathcal{O}_X$-modules. It follows that its hypercohomology, as well that of $\mathcal{W}\Omega^{\cdot}/p\mathcal{W}\Omega^{\cdot}$, is finite dimensional. The exact sequences

$$0 \to \mathcal{W}\Omega^{\cdot}_X/p\mathcal{W}\Omega^{\cdot}_X \to \mathcal{W}\Omega^{\cdot}_X/p^{n+1}\mathcal{W}\Omega^{\cdot}_X \to \mathcal{W}\Omega^{\cdot}_X/p^n\mathcal{W}\Omega^{\cdot}_X \to 0$$

then allow us to conclude by induction that the cohomology of each $\mathcal{W}\Omega^{\cdot}/p^n\mathcal{W}\Omega^{\cdot}_X$ is a W-module of finite length. The same is true for $\mathcal{W}_n\Omega^{\cdot}_X$ by another application of Proposition 1.7. Then a Mittag-Leffler argument shows that the natural maps

$$H^*(X, \mathcal{W}\Omega^{\cdot}_X) \to \varprojlim H^*(X, \mathcal{W}_n\Omega^{\cdot}_X)$$

are isomorphisms. It follows that the cohomology modules are separated and complete for the p-adic topology. Since the terms of $\mathcal{W}\Omega^{\cdot}_X$ are p-torsion free, we find an inclusion

$$H^*(X, \mathcal{W}\Omega^{\cdot}_X)/pH^*(X, \mathcal{W}\Omega^{\cdot}_X) \to H^*(X, \mathcal{W}\Omega^{\cdot}_X/p\mathcal{W}\Omega^{\cdot}_X).$$

Since the latter is finitely generated, so is the former, and it follows that the same is true of $H^*(X, \mathcal{W}\Omega^{\cdot}_X)$. ☐

Remark 4.4. In fact, as explained in [12, §6.2], when X/k is proper, the coherence of the sheaves $\mathcal{W}_1\Omega^j_X$ is enough to establish that $R\Gamma(X, \mathcal{W}\Omega^{\cdot}_X)$ is finitely generated over the Raynaud ring.

If X/k is of finite type and normal, then its smooth locus $X_{\rm sm}$ is open and its complement has codimension at least two. If $j\colon X_{\rm sm} \to X$ is the inclusion, then the "Zariski differentials" of X/k are by definition the sheaves

$$\tilde{\Omega}^{\cdot}_{X/k} := j_*\Omega^{\cdot}_{X_{\rm sm}/k}.$$

These sheaves have been extensively studied [5], [7]; in particular Danilov has shown that if $k = \mathbf{C}$ and X has at most toroidal singularities, then the hypercohomology of $j_*\Omega^{\cdot}_{X_{\rm sm}/\mathbf{C}}$ is isomorphic to the singular cohomology of the analytic space associated to X. Thus it is natural to ask whether, when k is perfect of characteristic p and X has at most toroidal singularities, the natural map

$$\mathcal{W}_n\Omega^{\cdot}_X \to j_*\mathcal{W}_n\Omega^{\cdot}_{X_{\rm sm}/\mathbf{C}}$$

is an isomorphism. Unfortunately, this is not always the case, as explained in the appendix. On the other hand, as Danilov has explained in [6, Lemma 1.5], if X/k has at most toroidal singularities, and if $f\colon X' \to X$ is any resolution of singularities, then there is a natural isomorphism

$$f_*\Omega^{\cdot}_{X'/\mathbf{C}} \to j_*\Omega_{X_{\rm sm}/\mathbf{C}}$$

It turns out the natural analog of this statement for the saturated de Rham Witt complex does hold. Since we do not know resolution of singularities in general, our statement is somewhat ad hoc. To make sense of it, let us say that a morphism $f\colon X' \to X$ is a "toroidal blowup" if, étale locally on X, there exists a toric monoid Q and an ideal K of Q such that $X = \operatorname{Spec} k[Q]$ and $X' \to X$ is the normalized blowup of X along the ideal $k[K]$. In this case both X and X' have at most toroidal singularities.

Theorem 4.5. *Suppose that X/k has at most toroidal singularities and that $f\colon X' \to X$ is a toroidal blowup.*

1. *The natural map $\mathcal{W}_n\Omega^{\cdot}_X \to f_*(\mathcal{W}_n\Omega^{\cdot}_{X'})$ is an isomorphism for all n.*
2. *If X'/k is smooth, the natural maps*

$$\mathcal{W}_n\Omega^{\cdot}_X \to f_*(W_n\Omega^{\cdot}_{X'}) \text{ and } \mathcal{W}_1\Omega^{\cdot}_X \to f_*(\Omega^{\cdot}_{X'/k})$$

are isomorphisms.

Proof. Statement (1) implies statement (2), since $\mathcal{W}_n\Omega^{\cdot}_{X'}$ is naturally isomorphic to $W_n\Omega^{\cdot}_{X'}$ when X'/k is smooth. Statement (1) can be verified étale locally on X, so we may and shall assume that Q is a toric monoid, that $X = \operatorname{Spec} k[Q]$, and that $X' \to X$ is the normalized blowup of X along the ideal of $k[Q]$ generated by an ideal K of Q. Statement (3) of Theorem 3.4 provides natural isomorphisms $\underline{\Omega}^i_{X/k} \cong \mathcal{W}_1\Omega^i_{X/k}$ and $\underline{\Omega}^i_{X'/k} \cong \mathcal{W}_1\Omega^i_{X'/k}$. Then Proposition A.2 of the appendix shows that the theorem is true when $n = 1$. We proceed by induction on n, using some results of Illusie and the following lemma, whose proof is immediate.

Lemma 4.6. *Let $0 \to A \to B \to C$ (resp. $0 \to A' \to B' \to C'$) be an exact sequence of abelian sheaves on X (resp. X') and that there exists a commutative diagram as follows.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "a"] & B \arrow[r, "\pi"] \arrow[d, "b"] & C \arrow[d, "c"] \\
0 \arrow[r] & f_*A' \arrow[r] & f_*B' \arrow[r, "\pi'"] & f_*C'
\end{tikzcd}$$

1. *If b and c are isomorphisms, so is a.*
2. *If a and c are isomorphisms and π is an epimorphism, then π' is an epimorphism and b is an isomorphism.* □

Applying the induction hypothesis and statement (2) of this lemma to the exact sequence:

$$0 \longrightarrow \operatorname{Gr}^n_{Fil} \mathcal{W}\Omega^i_X \longrightarrow \mathcal{W}_{n+1}\Omega^i_X \xrightarrow{\pi} \mathcal{W}_n\Omega^i_X \longrightarrow 0,$$

we see that it will be enough to prove that the natural map

$$g_n\colon\ \operatorname{Gr}^n_{Fil} \mathcal{W}\Omega^i_X \to \operatorname{Gr}^n_{Fil} \mathcal{W}\Omega^i_{X'}$$

is an isomorphism for every n. To do this, we use Illusie's exact sequences [12, §6.2]:

$$0 \longrightarrow \mathcal{W}_1\Omega^i_X/B_n\mathcal{W}_1\Omega^i_X \longrightarrow \operatorname{Gr}^n_{Fil}\mathcal{W}\Omega^i_X \longrightarrow \mathcal{W}_1\Omega^i_X/Z^n\mathcal{W}_1\Omega^i_X \longrightarrow 0.$$

Applying (2) of Lemma 4.6, we see that it will suffice to show that the maps

$$\begin{array}{lcl} z_n\colon \mathcal{W}_1\Omega^i_X/Z^n & \longrightarrow & f_*(\mathcal{W}_1\Omega^i_{X'}/Z^n) \\ b_n\colon \mathcal{W}_1\Omega^i_X/B_n & \longrightarrow & f_*(\mathcal{W}_1\Omega^i_{X'}/B_n) \end{array}$$

are isomorphisms. We recall from [11, 2.2] that $Z^{\cdot}$ and $B.$ are, respectively, the "iterated cycle" and "iterated boundary" filtrations defined inductively using the Cartier isomorphisms (1.7) $\psi_1\colon \mathcal{W}_1\Omega^i_X \cong H^i(\mathcal{W}_1\Omega^{\cdot}_X)$ by:

$$\begin{array}{lcl} B_0\mathcal{W}_1\Omega^i_X & := & 0 \\ B_1\mathcal{W}_1\Omega^i_X & := & \operatorname{Im}(d\colon \mathcal{W}_1\Omega^{i-1}_X \to \mathcal{W}_1\Omega^i_X) \\ B_n\mathcal{W}_1\Omega^i_X & \xrightarrow[\cong]{\beta_{i,n}} & B_{n+1}\mathcal{W}_1\Omega^i_X/B_1\mathcal{W}_1\Omega^i_X \\ Z^0\mathcal{W}_1\Omega^i_X & := & \mathcal{W}_1\Omega^i_X \\ Z^1\mathcal{W}_1\Omega^i_X & := & \operatorname{Ker}(d\colon \mathcal{W}_1\Omega^i_X \to \mathcal{W}_1\Omega^{i+1}_X) \\ Z^n\mathcal{W}_1\Omega^i_X & \xrightarrow[\cong]{\zeta_{i,n}} & Z^{n+1}\mathcal{W}_1\Omega^i_X/B_1\mathcal{W}_1\Omega^i_X. \end{array}$$

Note for future reference that for $n \geq 0$, $\zeta_{i,n}$ induces isomorphisms:

$$\operatorname{Gr}^n_Z \mathcal{W}_1\Omega^i_X \to \operatorname{Gr}^{n+1}_Z \mathcal{W}_1\Omega^i_X$$

and hence (by induction)

$$\operatorname{Gr}^n_Z \mathcal{W}_1\Omega^i_X \cong \operatorname{Gr}^0_Z \mathcal{W}_1\Omega^i_X \cong B_1\mathcal{W}_1\Omega^{i+1}_X \tag{4.1}$$

Similarly, combining $\zeta_{i,n}$ and $\beta_{i,n}$, we find isomorphisms:

$$Z^n\mathcal{W}_1\Omega^i_X/B_n\mathcal{W}_1\Omega^i_X \cong Z^{n+1}\mathcal{W}_1\Omega^i_X/B_{n+1}\mathcal{W}_1\Omega^i_X$$

and hence (by induction)

$$Z^n\mathcal{W}_1\Omega^i_X/B_n\mathcal{W}_1\Omega^i_X \cong \mathcal{W}_1\Omega^i_X. \tag{4.2}$$

We have already seen that the map

$$z_0\colon \mathcal{W}_1\Omega^i_X \to f_*(\mathcal{W}_1\Omega^i_{X'})$$

is an isomorphism. Applying statement (1) of Lemma 4.6 to the exact sequence

$$0 \to Z^1\mathcal{W}_1\Omega^i_X \longrightarrow \mathcal{W}_1\Omega^i_X \xrightarrow{d} \mathcal{W}_1\Omega^{i+1}_X,$$

we see that the map

$$Z^1\mathcal{W}_1\Omega^i_X \to f_*(Z^1\mathcal{W}_1\Omega^i_{X'})$$

is an isomorphism. Thanks to the Cartier isomorphism on X and X' and the case $n = 1$, we also know that the map

$$\mathcal{W}\Omega^i_{X/k} \cong \mathcal{H}^i(\mathcal{W}_1\Omega^{\cdot}_X) \to f_*(\mathcal{H}^i(\mathcal{W}_1\Omega^i_{X'})) \cong f_*(\mathcal{W}_1\Omega^i_{X'/k})$$

is an isomorphism. Applying (1) of Lemma 4.6 to the exact sequence

$$0 \longrightarrow B_1\mathcal{W}_1\Omega^i_X \to Z^1\mathcal{W}_1\Omega^i_X \longrightarrow \mathcal{H}^i(\mathcal{W}_1\Omega^i_X) \longrightarrow 0,$$

we deduce that the map

$$B_1\mathcal{W}_1\Omega^i_X \to f_*(B_1\mathcal{W}_1\Omega^i_{X'})$$

is an isomorphism for all i. Equation 4.1 then implies that the map

$$\operatorname{Gr}^n_Z \mathcal{W}_1\Omega^i_X \to f_*(\operatorname{Gr}^n_Z \mathcal{W}_1\Omega^i_{X'})$$

is also an isomorphism. Applying (2) of Lemma 4.6 to the exact sequence

$$0 \longrightarrow \operatorname{Gr}^n_Z \mathcal{W}_1\Omega^i_X \longrightarrow \mathcal{W}_1\Omega^i_X/Z^{n+1}\mathcal{W}_1\Omega^i_X \longrightarrow \mathcal{W}_1\Omega^i_X/Z^n\mathcal{W}_1\Omega^i_X \longrightarrow 0,$$

we see by induction on n that

$$z_n\colon \operatorname{Gr}^n_Z \mathcal{W}_1\Omega^i_X \to f_*(\operatorname{Gr}^n_Z \mathcal{W}_1\Omega^i_{X'})$$

is an isomorphism for all n.

It remains only to prove that b_n is an isomorphism. Using equation 4.2, we find an exact sequence:

$$0 \longrightarrow B_n\mathcal{W}_1\Omega^i_X \longrightarrow Z^n\mathcal{W}_1\Omega^i_X \longrightarrow \mathcal{W}_1\Omega^i_X \longrightarrow 0.$$

Applying (2) of Lemma 4.6, we can conclude that the map

$$b_n\colon \mathcal{W}_1\Omega^i_X/B_n \to f_*(\mathcal{W}_1\Omega^i_{X'}/B_n)$$

is indeed an isomorphism, concluding the proof. □

5 Crystalline cohomology

The construction of the saturated de Rham Witt complex of an $\mathbf{F}_p$-scheme X explained in Proposition 1.12 depends on the existence of a lifting of X together with a lifting of its Frobenius endomorphism. The existence of such liftings is rare, but often one can find an embedding of X as a closed subscheme of a scheme which does admit such a lifting. We shall see that applying the construction in Proposition 1.12 to the PD-envelope of X in such a lifting gives another construction of $\mathcal{W}\Omega^{\cdot}_X$ and provides a direct way to compare de Rham-Witt and crystalline cohomology. Before explaining how this works, we need to control the torsion in such PD-envelopes.

Lemma 5.1. *If X is a reduced scheme of finite type over a perfect field k, embedded as a closed subscheme of a smooth formal scheme Y/W endowed with a Frobenius lift, let $D_X(Y)$ denote the (p-adically completed) PD-envelope of X in Y and let $\mathcal{O}_D$ denote the structure sheaf of $D_X(Y)$. Then the p-torsion of $\mathcal{O}_D$ forms a sub PD-ideal of the PD-ideal $\overline{\mathcal{I}}_D$ of X in $D_X(Y)$, as does its closure in the p-adic topology.*

Proof. Let us use affine notation for simplicity. We suppose $X = \operatorname{Spec} A$ and $Y = \operatorname{Spf} B$. Let A^{perf} denote the perfection of A. Since A is reduced, the map $A \to A^{\mathrm{perf}}$ is injective, and since A^{perf} is perfect, its Witt ring $W(A^{\mathrm{perf}})$ is p-torsion free. Hence the same is true of $W(A)$. The lift ϕ of Frobenius gives B the structure of a δ-ring, and by Joyal's characterization [13] of the functor W, there is a unique homomorphism $B \to W(A)$ of δ-rings which is compatible with the given map $B \to A$. Since the ideal of A in $W(A)$ has a canonical divided power structure, this map extends to a PD-homomorphism $D_I(B) \to W(A)$. Since $W(A)$ is p-torsion free, the p-torsion of $D_I(B)$ maps to zero in $W(A)$, hence also in A, and hence is contained in $\overline{I}$. Say $x \in \overline{I}$ and $p^r x = 0$. Then $p^{ir}\gamma_i(x) = \gamma_i(p^r x) = 0$, so $\gamma_i(x)$ is also a p-torsion element. This shows that the p-torsion of $\mathcal{O}_D$ forms a sub PD-ideal J of $\overline{I}$. Since $\overline{I}$ is p-adically closed, the closure of J is also contained in $\overline{I}$, and since the divided power operations γ_i are p-adically continuous, this closure is stable under their action. □

Suppose now that X/k is of finite type and reduced, embedded as a locally closed subscheme of a smooth formal scheme Y/W endowed with a Frobenius lift ϕ_Y, Let $\tilde{D}$ denote the closed subscheme of $D := D_X(Y)$ defined by the ideal of p-torsion elements of the structure sheaf $\mathcal{O}_D$ of $D_X(Y)$. It follows from Lemma 5.1 that the embedding $X \to D_X(Y)$ factors through $\tilde{D}$, and that the ideal of X in $\tilde{D}$ (which we denote by $\tilde{I}$) is again a PD-ideal. The $\mathcal{O}_Y$-module $\mathcal{O}_{D_X(Y)}$ admits an integrable connection [3, 6.4] which induces a connection on $\mathcal{O}_{\tilde{D}}$. The endomorphism ϕ_Y of Y extends uniquely to a PD-morphism ϕ_D of $D_X(Y)$, which in turn induces endomorphism, $\phi_{\tilde{D}}$ of $\tilde{D}$, of the module with connection $(\mathcal{O}_{\tilde{D}}, \nabla)$, and of its de Rham complex $\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}$. This endomorphism is divisible by p^i on $\mathcal{O}_{\tilde{D}} \otimes \Omega^i_{Y/W}$, so $\phi^i_{\tilde{D}} = p^i F$ for a unique $\mathcal{O}_{\tilde{D}}$-linear endomorphism F of $\mathcal{O}_{\tilde{D}} \otimes \Omega^i_{Y/W}$. Thus $(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}, d, F)$ is a Dieudonné complex.

Theorem 5.2. *With the notation of the previous paragraph, let $\tilde{D}_1$ denote the reduction of $\tilde{D}$ modulo p.*

1. *The Dieudonné complex $(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}, d, F)$ is in fact a Dieudonné algebra.*

2. *The natural map $\mathcal{O}_{\tilde{D}_1} \to \mathcal{W}_1 Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}, d, F)^0$ factors through a map*
$$\mathcal{O}_X \to \mathcal{W}_1 Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}, d, F)^0.$$

3. *The adjoint to the map in (2) is an isomorphism*
$$(\mathcal{W}\Omega^{\cdot}_X, d, F) \to \mathcal{W} Sat(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}, d, F)$$

Proof. To see that $(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}, d, F)$ is a Dieudonné algebra, we must show that $\phi_{\tilde{D}}\colon \tilde{D} \to \tilde{D}$ reduces to the Frobenius endomorphism of $\tilde{D}_1$ [4, 3.1.2]. Since $\tilde{D}_1 \subseteq D_1$, It will suffice to show that ϕ_D reduces to the Frobenius endomorphism F_{D_1} of D_1. By definition, ϕ_D is the unique PD morphism $D \to D$ extending F_Y, and so it will suffice to show that F_{D_1} is in fact a PD-morphism. But if t is an element of the PD-ideal $\overline{I}$ of X in D_1, then $F^*_{D_1}(t) = t^p = p!t^{[p]} = 0$, and hence for any $n \geq 1$, $F_{D_1} \circ \gamma_n$ and $\gamma_n \circ F_{D_1}$ both vanish.

Note that $\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}$ is not the same as the de Rham complex of $\tilde{D}$; the latter has a lot of p-torsion. The comparison of these two complexes is the key to our proof.

Lemma 5.3. *In the following diagram, the top horizontal arrow is induced by adjunction and the natural map $\Omega^1_{Y/W} \to \Omega^1_{\tilde{D}/Y}$, and $\tilde{\nabla} := \pi \circ t \circ \nabla$. The lower triangle commutes, but the upper one does not.*

$$\begin{tikzcd}
\mathcal{O}_{\tilde{D}} \otimes \Omega^1_{Y/W} \arrow[r, "t"] & \Omega^1_{\tilde{D}/W} \arrow[d, "\pi"] \\
\mathcal{O}_{\tilde{D}} \arrow[u, "\nabla"] \arrow[ur, "d"] \arrow[r, "\tilde{\nabla}"] & \Omega^1_{\tilde{D}/W}/(p\text{-torsion})^-
\end{tikzcd}$$

Furthermore, the composite

$$\tilde{t} := \pi \circ t\colon \mathcal{O}_{\tilde{D}} \otimes \Omega^1_{Y/W} \longrightarrow \Omega^1_{\tilde{D}/W}/(p\text{-torsion})^-$$

is an isomorphism. (NB: here we always mean the p-adically completed de Rham complexes; and in particular we are dividing by the p-adic closure of the p-torsion in the lower right hand corner.)

Proof. The algebra $\mathcal{O}_{\tilde{D}}$ is topologically generated over $\mathcal{O}_Y$ by the divided powers $f^{[n]}$ of elements f of the ideal of X in Y, for $n \geq 1$. For any such f, we have $\nabla f^{[n]} = f^{[n-1]} \otimes df$ in $\mathcal{O}_{\tilde{D}} \otimes \Omega^1_{Y/W}$ [3, 6.4]. On the other hand, since $n!f^{[n]} = f^n$ and $n!f^{[n-1]} = nf^{n-1}$, we have

$$n!df^{[n]} = d(n!f^{[n]}) = df^n = nf^{n-1}df = n(n-1)!f^{[n-1]}df = n!f^{[n-1]}df$$

in $\Omega^1_{\tilde{D}/W}$. Thus $\nabla f^{[n]}$ and $df^{[n]}$ have the same image in $\Omega^1_{\tilde{D}/W}/(p\text{-torsion})^-$, so the lower triangle commutes.

Since $d\colon \mathcal{O}_{\tilde{D}} \to \Omega^1_{\tilde{D}/W}$ is the universal derivation to a p-adically complete sheaf of $\mathcal{O}_{\tilde{D}}$-modules, there is a unique map $s\colon \Omega^1_{\tilde{D}/W} \to \mathcal{O}_{\tilde{D}} \otimes \Omega^1_Y$ such that $s \circ d = \nabla$. The map s factors through a map $\tilde{s}\colon \Omega^1_{\tilde{D}/W}/(p\text{-torsion})^- \to \mathcal{O}_{\tilde{D}} \otimes \Omega^1_Y$.

We find the diagram:

$$\begin{tikzcd}
 & \Omega^1_{\tilde{D}/W}/(p\text{-torsion})^- \arrow[r, "\tilde{s}"] & \mathcal{O}_{\tilde{D}} \otimes \Omega^1_Y \arrow[r, "\tilde{t}"] & \Omega^1_{\tilde{D}/W}/(p\text{-torsion})^- \\
\Omega^1_{Y/W} \arrow[r] & \Omega^1_{\tilde{D}/W} \arrow[u, "\pi"] \arrow[r, "s"] & \mathcal{O}_{\tilde{D}} \otimes \Omega^1_{Y/W} \arrow[u, "\mathrm{id}"] \arrow[r, "t"] & \Omega^1_{\tilde{D}/W} \arrow[u, "\pi"] \arrow[d, "\pi"] \\
 & \mathcal{O}_Y \arrow[ul] \arrow[u] \arrow[r, "i"] & \mathcal{O}_{\tilde{D}} \arrow[ul, "d"] \arrow[u, "\nabla"] \arrow[ur, "d"] \arrow[r, "\tilde{\nabla}"] & \Omega^1_{\tilde{D}/W}/(p\text{-torsion})^-
\end{tikzcd}$$

in which all triangles commute, except for the upper one in the right-hand bottom square.

Then $\pi \circ t \circ s \circ d = \pi \circ t \circ \nabla = \tilde{\nabla} = \pi \circ d$, and it follows that $\pi \circ t \circ s = \pi$ and hence that $\tilde{t} \circ \tilde{s} = \mathrm{id}$. On the other hand, if f is a local section of $\mathcal{O}_Y$, then $i(f)$ is a section of $\mathcal{O}_{\tilde{D}}$, and $\nabla(i(f)) = 1 \otimes df$ in $\mathcal{O}_{\tilde{D}} \otimes \Omega^1_{Y/W}$. Thus the problematic triangle does commute when restricted to $\mathcal{O}_Y$; that is, $t \circ \nabla \circ i = d \circ i$. It follows that $s \circ t\colon \mathcal{O}_{\tilde{D}} \otimes \Omega^1_{Y/W} \to \mathcal{O}_{\tilde{D}} \otimes \Omega^1_{Y/W} = \mathrm{id}$, and hence the same is true of $\tilde{s} \circ \tilde{t}$. $\square$

Since $(\tilde{D}, \phi_{\tilde{D}})$ is a p-torsion free lifting of $(\tilde{D}_1, F_{\tilde{D}_1})$, by [4, 3.2.1] there is an endomorphism F of the graded abelian sheaf $\Omega^{\cdot}_{\tilde{D}/W}$ which gives it the structure of a Dieudonné algebra, and [4, 4.2.3] constructs an isomorphism of Dieudonné algebras:

$$\mathcal{W}\,\mathrm{Sat}(\Omega^{\cdot}_{\tilde{D}/W}, d, F) \to (\mathcal{W}\Omega^{\cdot}_{\tilde{D}_1}, d.F).$$

Thus we find a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W} \arrow[r, "\pi \circ t"] \arrow[d] & \Omega^{\cdot}_{\tilde{D}/W}/(p\text{-torsion})^- \arrow[d] & \\
\mathcal{W}\,\mathrm{Sat}(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}) \arrow[r, "w"] & \mathcal{W}\,\mathrm{Sat}(\Omega^{\cdot}_{\tilde{D}/W}) \arrow[r, "\cong"] & \mathcal{W}\Omega^{\cdot}_{\tilde{D}_1} \arrow[dl, "g"] \\
 & \mathcal{W}\Omega^{\cdot}_X &
\end{tikzcd}$$

We have seen in Lemma 5.3 that $\pi \circ t$ is an isomorphism, and hence the same is true of w. Since X is the reduced subscheme of $\tilde{D}_1$, the map g is also an isomorphism [4, 6.5.2]. We conclude that the natural map $\mathcal{W}\,\mathrm{Sat}(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}) \to \mathcal{W}\Omega^{\cdot}_X$ is an isomorphism, as asserted in the last statement of the theorem. $\square$

When X/k is smooth, we know from [4, 4.4.12] that $\mathcal{W}\Omega^{\cdot}_X$ agrees with the classical de Rham Witt complex $W\Omega^{\cdot}_X$, which is known to compute crystalline cohomology [11, II.1.4]. The previous result gives a new and direct proof of this fact.

Corollary 5.4. *Suppose that X/k is smooth and embedded as a locally closed subscheme of a smooth formal scheme Y/W which is endowed with a lifting ϕ_Y of the Frobenius endomorphism of its reduction p. Then the map*

$$c\colon (\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}, d) \to \mathcal{W}Sat(\mathcal{O}_{\mathcal{O}_{\tilde{D}}} \otimes \Omega^{\cdot}_{Y/W}, d) \cong \mathcal{W}\Omega^{\cdot}_X$$

is a quasi-isomorphism Thus, $(\mathcal{W}\Omega^{\cdot}_X, d)$ is a representative of $Ru_{X/W*}(\mathcal{O}_{X/W})$.

Proof. By [3, 8.20], applied to the constant gauge $\epsilon = 0$, the morphism $\phi^{\cdot}_{\tilde{D}}$ factors through a quasi-isomorphism

$$\alpha\colon \mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W} \to \eta(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}).$$

Thus the complex $(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}, d, F)$ is quasi-saturated (1.4), and so Theorem 1.8 implies that the map

$$(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}, d, F) \to \mathcal{W}\,\mathrm{Sat}(\mathcal{O}_{\tilde{D}} \otimes \Omega^{\cdot}_{Y/W}, d, F)$$

is a quasi-isomorphism. $\square$

Remark 5.5. In fact, there is a more direct proof of Corollary 5.4, which does not refer to [3] or to statement (3) of Theorem 5.2. We explain this in a more general context in the proof of Theorem 6.8 in the next section.

6 Log schemes

In the context of log geometry, one can define, in a somewhat ad hoc way, a variant of crystalline cohomology that coincides with the saturated de Rham-Witt cohomology we have been considering. This construction will allow us to obtain more precise information about the action of Frobenius, about the behaviour of the Hodge and conjugate spectral sequences, and about the relationship between the de Rham-Witt complex and the de Rham cohomology of a lifting. The log structures do not play a role in the construction of the de Rham-Witt complex we are considering here, but they seem to be important in the construction the crystalline complexes and in controlling the liftings from characteristic p to characteristic zero.

Let us first explain why the results of section 4 will be relevant to our constructions here. If Q is a monoid and R is a ring (understood from the context), we denote by A_Q the log scheme $\mathrm{Spec}(Q \to R[Q])$, and if K is an ideal Q, we denote by $\mathsf{A}_{\mathsf{Q},\mathsf{K}}$ the idealized log scheme $\mathrm{Spec}((Q, K) \to R[Q, K])$ [20, III,§1.3]. If X is a log scheme, we denote by $\underline{X}$ the underlying scheme, which can also be viewed as a log scheme with trivial log structure.

Proposition 6.1. *Let R be a ring (with no log structure) and let Y/R be a fine saturated and smooth idealized log scheme over R. Then étale locally on Y, there exist a toric monoid Q, an ideal K in Q, and a strict étale morphism $Y \to \mathsf{A}_{\mathsf{Q},\mathsf{K}}$. In particular, if R is a field, then Y is ideally toroidal in the sense of §4.*

Proof. If $\overline{y}$ is a geometric point of Y, the stalk $\overline{\mathcal{M}}_{Y,\overline{y}}$ is a fine, saturated, and sharp, hence toric monoid [20, I,1.3.5], and in some neighborhood X of $\overline{y}$ there exists a chart $(Q, K) \to (\mathcal{M}_Y, \mathcal{K}_Y)$ inducing an isomorphism $(Q, K) \to (\overline{\mathcal{M}}_{\overline{y}}, \mathcal{K}_y)$ such that the associated morphism $\underline{X} \to \mathrm{Spec}(R[Q]/R[K])$ is étale [20, IV,3.3.4,3.3.5]. ☐

We begin with a discussion of de Rham cohomology. Let T be a scheme (with trivial log structure), and let Y/T be a smooth, fine, and saturated idealized log scheme over T. We denote by $\Omega^{\cdot}_{Y/T}$ the *logarithmic* de Rham complex of Y/T [20]; in particular, $(d, dlog)\colon (\mathcal{O}_Y, \mathcal{M}_Y) \to \Omega^1_{Y/T}$ is the universal log derivation. When $T = \mathrm{Spec}(\mathbf{C})$ and $K = \emptyset$, this complex calculates the cohomology of Y^*, the open subset of Y where its log structure is trivial [20, V,4.2.5]. As explained in [20, V,2.3.21], the complex $\Omega^{\cdot}_{Y/T}$ has a canonical subcomplex $\underline{\Omega}^{\cdot}_{Y/T}$ with the following properties.

1. If K is an ideal in a toric monoid Q, if $T = \mathrm{Spec}(R)$, and if Y is the log scheme $\mathsf{A}_{\mathsf{Q},\mathsf{K}}$, then $\underline{\Omega}^{\cdot}_{Y/T}$ is the complex of sheaves corresponding to the complex $\underline{\Omega}^{\cdot}_{R[Q,K]}$ from Definition 2.3 (and the discussion at the beginning of §3 for the idealized case).

2. If $f\colon Y' \to Y$ is a strict and étale morphism of idealized fs log schemes, the natural map $f^*\underline{\Omega}^{\cdot}_{Y/T} \to \underline{\Omega}^{\cdot}_{Y'/T}$ is an isomorphism. (A morphism $f\colon Y' \to Y$ of idealized log schemes is said to be *strict* if the map $f^\flat\colon f^*M_Y \to M_{Y'}$ is an isomorphism and the ideal K_Y generates the ideal $K_{Y'}$.)

3. Formation of $\underline{\Omega}^{\cdot}_{Y/T}$ is compatible with arbitrary base change $T' \to T$.

We do not know how to define such complexes for general schemes with toroidal singularities without the additional information provided by a global log structure. [2]

Remark 6.2. Before proceeding, a word of warning. If $f\colon Y \to Z$ is a (log) étale morphism of log schemes, the natural map $f^*\Omega^{\cdot}_{Z/T} \to \Omega^{\cdot}_{Y/T}$ is an isomorphism, but this need not be true for the map $f^*\underline{\Omega}^{\cdot}_{Z/T} \to \underline{\Omega}^{\cdot}_{Y/T}$ unless f is also strict. In particular, recall that, if $X \to Y$ is a closed immersion of log schemes the *strict* formal completion $\hat{Y}$ of Y along X is constructed as follows. First one forms the usual formal completion $\hat{Y}'$ of Y along X, then the exactification $\hat{Y}'' \to \hat{Y}'$ with respect to the morphism $X \to \hat{Y}'$, so that $X \to \hat{Y}''$ is strict and

[2] A more general construction appears in [20], where a subcomplex of $\Omega^{\cdot}_{Y/T}$ is is constructed associated to any relatively coherent sheaf $\mathcal{F}$ of faces of $\mathcal{M}_Y$. Here we only consider the case in which $\mathcal{F} = \mathcal{M}^*_Y$.

$\overline{M}_{\hat{Y}''} \to \overline{M}_X$ is an isomorphism. Then the ideal $K_X \subseteq M_X$ generates an ideal $K \subseteq M_{\hat{Y}''}$, and $\hat{Y}$ is the closed formal subscheme of $\hat{Y}''$ defined by $\alpha_{\hat{Y}''}(K)$. The complexes $\Omega^{\cdot}_{\hat{Y}'/T}$, $\Omega^{\cdot}_{\hat{Y}''/T}$, and $\Omega^{\cdot}_{\hat{Y}/T}$ can be identified with the respective pullbacks of $\Omega^{\cdot}_{Y/T}$, but this is not the case for their underlined counterparts. We shall need to be wary of this fact when forming (strict) PD envelopes.

In this rest of this section, we let X/k be a fine, saturated, reduced, and smooth idealized log scheme over a perfect field k of characteristic p. The following result generalizes the comparison between the de Rham and the de Rham-Witt complexes of a smooth scheme to the logarithmic case. It implies that for the log schemes schemes we are considering, the complex $\underline{\Omega}^{\cdot}_{X/k}$ does not depend on the log structure.

Theorem 6.3. *If X/k is a reduced and smooth idealized fs log scheme, there is a natural isomorphism*

$$\underline{\Omega}^{\cdot}_{X/k} \cong \mathcal{W}_1\Omega^{\cdot}_{\underline{X}},$$

uniquely determined by its naturality and compatibility with the classical isomorphism in the case of schemes with trivial log structure.

Proof. We first consider the case in which the underlying scheme $\underline{X}$ is also smooth. Then the natural map $\Omega^{\cdot}_{\underline{X}/k} \to \mathcal{W}_1\Omega^{\cdot}_{\underline{X}}$ is an isomorphism. In fact the same is true for the natural map $\Omega^{\cdot}_{\underline{X}/k} \to \underline{\Omega}^{\cdot}_{X/k}$, as can be checked locally by reducing to the case in which $X = \mathsf{A}_{\mathsf{Q},\mathsf{K}}$. In this case, if $\underline{X}$ is smooth, then K is a prime ideal and its complement $Q \setminus K$ is a face G of Q, and necessarily G is a free monoid. It follows from the definitions that $\underline{\Omega}^{\cdot}_{X/k} \cong \underline{\Omega}^{\cdot}_{\mathsf{A}_\mathsf{G}/k}$, in which case the result follows from [20, V,2.3.11]. Then there is a unique isomorphism $\underline{\Omega}^{\cdot}_{X/k} \to \mathcal{W}_1\Omega^{\cdot}_X$ making the following diagram commute:

$$\begin{tikzcd}
\Omega^{\cdot}_{\underline{X}/k} \arrow[r, dashed] & \mathcal{W}_1\Omega^{\cdot}_{\underline{X}} \\
\underline{\Omega}^{\cdot}_{X/k} \arrow[u, "\cong"] \arrow[ur, "\cong"] &
\end{tikzcd}$$

In general, if X/k is a smooth saturated and reduced idealized log scheme, the set X_{sm} where $\underline{X}$ is smooth is open and dense, and we have the solid arrows in the diagram:

$$\begin{tikzcd}
\mathcal{W}_1\underline{\Omega}^{\cdot}_X \arrow[r] & j_*(\mathcal{W}_1\Omega^{\cdot}_{X_{\text{sm}}}) \arrow[d, "\cong"] \\
\Omega^{\cdot}_{\underline{X}/k} \arrow[u, dashed] \arrow[r] & j_*(\underline{\Omega}^{\cdot}_{X\text{sm}/k}).
\end{tikzcd}$$

If X admits an étale chart to some $\mathsf{A}_{\mathsf{Q,K}}$, statement (3) of Theorem 3.4 shows that the dashed arrow exists and is an isomorphism. This isomorphism is compatible with localization, and we see from Theorem 3.5 and the definitions (2.3) that the horizontal maps are injective. Thus, if there exists a global left vertical isomorphism making the diagram commute, it is unique. Since such an arrow does exist locally, the uniqueness guarantees that it also exists globally. $\square$

We now turn to our construction of a crystalline incarnation of the cohomology of the complexes $\underline{\Omega}^{\cdot}_Y$ we have been considering. Suppose that we are given a closed immersion of X into a fine, saturated, reduced, and smooth idealized log scheme Y/W. Let $\hat{Y}$ denote the strict formal completion of Y along X and let $(D_X(Y), \overline{J}_Y)$ be the (p-adically completed) strict PD-envelope of X in Y, or equivalently in $\hat{Y}$, and let $\mathcal{O}_D$ denote its structure sheaf. As Kato explained in [14], $\mathcal{O}_D$ is p-torsion free and has a canonical integrable connection ∇ whose de Rham complex $\mathcal{O}_D \otimes \Omega^{\cdot}_{Y/W}$ calculates the cohomology of the structure sheaf $\mathcal{O}_{X/W}$ of the (log) crystalline site of X/W. Since $X \to \hat{Y}$ is strict, ∇ factors through $\underline{\Omega}^{\cdot}_{\hat{Y}/W}$ and thus defines a subcomplex $\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{\hat{Y}/W}$ of $\mathcal{O}_D \otimes \Omega^{\cdot}_{\hat{Y}/W}$. These complexes can be endowed with an analog of the Hodge filtration, and we shall see that the resulting cohomology is crystalline in nature. We need to prepare with some technicalities concerning these filtered complexes.

Proposition 6.4. *If $X \to Y$ is a closed immersion as described in the previous paragraph, let*

$$
\begin{aligned}
Fil^k_X \Omega^i_{\hat{Y}/W} &:= \overline{J}^{[k-i]}_{\hat{Y}} \otimes \Omega^i_{\hat{Y}/W} \\
Fil^k_X \underline{\Omega}^i_{\hat{Y}/W} &:= \overline{J}^{[k-i]}_{\hat{Y}} \otimes \underline{\Omega}^i_{\hat{Y}/W}
\end{aligned}
$$

Then in fact

$$
Fil^k_X \underline{\Omega}^i_{\hat{Y}/W} = Fil^k_X \Omega^{\cdot}_{\hat{Y}/W} \cap \underline{\Omega}^i_{\hat{Y}/W},
$$

so that $(\underline{\Omega}^{\cdot}_{\hat{Y}/W}, Fil^{\cdot}_X)$ is a strict filtered subcomplex of $(\Omega^{\cdot}_{\hat{Y}/W}, Fil^{\cdot}_X)$.

Proof. The proof will rely on two lemmas, the second of which is formulated somewhat more generally than we will need here.

Lemma 6.5. *Let $f\colon Y \to T$ be a smooth morphism of idealized fs log formal schemes, where T has trivial log structure. Then the sheaves $\underline{\Omega}^i_{Y/T}$, as well as the quotients $\Omega^i_{Y/T}/\underline{\Omega}^i_{Y/T}$, are flat over T, and their formation commutes with base change $T' \to T$. If f admits a factorization $f = p \circ h$, where $h\colon Y \to Z$ is smooth and $p\colon Z \to T$ is smooth and strict, then the sheaves $\underline{\Omega}^i_{Y/T}$ and $\Omega^i_{Y/T}/\underline{\Omega}^i_{Y/T}$ are also flat over Z.*

Proof. We can check these statements étale locally, and so, by Proposition 6.1, we may assume that $T = \mathrm{Spec}(R)$ and that $Y = \mathrm{Spec}((Q,K) \to R[Q,K])$, where (Q,K) is a fine saturated idealized monoid. For each $q \in Q$, the group $\langle q\rangle^{\mathrm{gp}}$ is free abelian and a direct summand of the free abelian Q^{gp}, and hence each $R\otimes\Lambda^i\langle q\rangle^{\mathrm{gp}}$ is free and a direct summand of the free R-module $R\otimes\Lambda^i Q^{\mathrm{gp}}$. Since $\underline{\Omega}^i_{R[Q,K]/R}\cong\oplus_{q\notin K}R\otimes\Lambda^i\langle q\rangle^{\mathrm{gp}}$, it is a free R-module. Similarly,

$$\Omega^i_{R[Q,K]/R}/\underline{\Omega}^i_{R[Q,K]/R}=\bigoplus_{q\notin K}R\otimes\Lambda^i Q^{\mathrm{gp}}/\Lambda^i\langle q\rangle^{\mathrm{gp}}$$

is a direct sum of free R-modules, hence free. This proves the flatness, and since both $\Omega^i_{Y/T}$ and $\underline{\Omega}^i_{Y/T}$ commute with base change, the same is true of their quotient.

As a first step toward the second statement, we shall show that $\underline{\Omega}^i_{R[Q,K]/R}$ and $\Omega^i_{R[Q,K]/R}/\underline{\Omega}^i_{R[Q/K]/R}$ are free over $R[Q^*]$. Since Q is saturated, it can be written as a product $Q=Q^*\oplus\overline{Q}$. Having chosen a section $\overline{Q}\to Q$, we can thus write every element of q uniquely as a sum $q=\overline{q}+u$, with $u\in Q^*$ and $\overline{q}\in\overline{Q}\subseteq Q$. Then $\langle q\rangle=\langle\overline{q}\rangle\subseteq Q$, and we see that there are isomorphisms of $R[Q^*]$-modules:

$$\begin{aligned}
\underline{\Omega}^i_{R[Q,K]/R} &\cong R[Q^*]\otimes\bigoplus_{\overline{q}\in\overline{Q}\setminus\overline{K}}\Lambda^i\langle\overline{q}\rangle^{\mathrm{gp}}\\
\Omega^i_{R[Q,K]/R}/\underline{\Omega}^i_{R[Q,K]/R} &\cong R[Q^*]\otimes\bigoplus_{\overline{q}\in\overline{Q}\setminus\overline{K}}\Lambda^i Q^{\mathrm{gp}}/\Lambda^i\langle\overline{q}\rangle^{\mathrm{gp}}
\end{aligned}$$

Thus both of these $R[Q^*]$-modules are free. Now to prove the second statement, working étale locally, we may assume that $T=\operatorname{Spec}R$ and that p is projection from affine n-space to T. After a further adjustment, we may in fact assume that p is the projection $\mathbf{G}_m^n\times T\to T$. Thus $Z=\operatorname{Spec}R[\Gamma]$, where Γ is a finitely generated free abelian group. By Proposition 6.1 applied to $Y/R[\Gamma]$, we may also assume that h admits a strict étale chart subordinate to an idealized toric monoid (P,K), and even that $Y=\operatorname{Spec}((P,K)\to R[\Gamma][P,K])$. Now let $Q:=P\oplus\Gamma$, so that $Q^*=P^*\oplus\Gamma$ and $Y=\operatorname{Spec}((Q,K\oplus\Gamma)\to R[Q,K\oplus\Gamma)$. The previous paragraph tells us that the modules under consideration are free over $R[Q^*]$, and it follows that they are flat over $R[\Gamma]$, as claimed. $\square$

Lemma 6.6. *Consider a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
S \arrow[r, "i"] & T,
\end{tikzcd}$$

where f and g are smooth integral morphisms of fs formal idealized log schemes and i and j are strict closed immersions. Then locally on Y and T, the mor-

phisms g and i factor:

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & Y \arrow[d, "h"] \\ S \arrow[r, "i'"] \arrow[dr, "i"] & Z := \mathbf{A}^r_T \arrow[d, "p"] \\ & T \end{tikzcd}$$

such that $p \circ h = g$, the square is Cartesian, the morphisms p and h are smooth and integral, and i' is a strict closed immersion. The analogous statement holds if Y is a formal idealized log scheme and X is a subscheme of definition, with Z replaced by formal affine space relative to T.

Proof. Let $\mathcal{J}$ be the ideal of X in Y, let $\mathcal{I}$ be the ideal of S in T, and let $Y_S := Y \times_T S$. Let $\mathcal{J}_S$ denote the ideal of X in Y_S, and consider the sequence of $\mathcal{O}_X$-modules:

$$0 \to (\mathcal{J}_S/\mathcal{J}_S^2) \to \Omega^1_{Y_S/S|_X} \to \Omega^1_{X/S} \to 0.$$

Since Y_S/S and X/S are smooth, this sequence is exact and locally split, and $(\mathcal{J}_S/\mathcal{J}_S^2)$ is locally free [20, IV, 3.2.2].[3] Working locally, we may choose sections $f_1, \ldots, f_r$ of $\mathcal{J}$ whose images form a basis for $(\mathcal{J}_S/\mathcal{J}_S^2)$. Let $Z := \mathbf{A}^r_T$, , let p be the structure map, let h be the map defined by $f_1, \ldots, f_r$, and let i' be the composition of i with the zero section of $\mathbf{A}^r_T$. Then the diagram shown commutes, and $p \circ h = g$. Since the images of $(f_1, \ldots, f_r)$ generate $\mathcal{J}_S$, the ideal $\mathcal{J}$ is generated by $(f_1, \ldots, f_r)$ and $\mathcal{I}$, and thus the square is Cartesian. The morphism i' is a strict closed immersion, and the morphism p is smooth, by construction. To see that h is smooth, consider the exact sequence

$$h^*\Omega^1_{Z/T} \to \Omega^1_{Y/T} \to \Omega^1_{Y/Z} \to 0.$$

The elements $(dt_1, \ldots, dt_r)$ of $\Omega^1_{Z/T}$ map to the elements $(df_1, \ldots, df_r)$ of $\Omega^1_{Y/T}$, and at each point x of X, these form part of a basis of the free $\mathcal{O}_{Y,x}$-module $\Omega^1_{Y/T,x}$. Thus the first map in the sequence is injective and locally split at x. Since g is smooth, it follows that h is also smooth at x [20, IV, 3.2.3], and hence in some neighborhood of x. Since p is strict, it is of course integral, and since $p \circ h$ is integral and p is strict, the morphism h is also integral [20, III,2.5.3]. The formal case is proved in the same way. $\square$

[3] Unfortunately this reference does not explicitly mention formal or idealized log schemes; however these variants present no difficulty.

We can now explain the proof of Proposition 6.4. Working locally, we apply Lemma 6.6, with f the map $X \to S := \operatorname{Spec} k$ and g the map $Y \to T := \operatorname{Spec} W$. The resulting map h is smooth and integral, so the underlying morphism $\underline{h}$ is flat [20, IV,4.3.5]. Since formation of divided power envelopes commutes with flat base change, $D_X(Y) \cong D_X(Z) \times_Z Y$. We have an exact sequence of $\mathcal{O}_Z$-modules

$$0 \to h_* \underline{\Omega}^i_{Y/T} \to h_* \Omega^i_{Y/T} \to h_*(\Omega^i_{Y/T}/\underline{\Omega}^i_{Y/T}) \to 0$$

and by Lemma 6.5, these are all flat. Then the sequence

$$0 \to h_* \underline{\Omega}^i_{Y/T} \to h_* \Omega^i_{Y/T} \to h_*(\Omega^i_{Y/T}/\underline{\Omega}^i_{Y/T}) \to 0$$

remains exact when tensored with $\mathcal{O}_{D_X(Z)}/\overline{J}_Z^{[k]}$, and the resulting sequence identifies with

$$\mathcal{O}_{D_X(Y)}/\overline{J}_Y^{[k]} \otimes \underline{\Omega}^i_{Y/T} \to \mathcal{O}_{D_X(Y)}/\overline{J}_Y^{[k]} \otimes \Omega^i_{Y/T} \to \mathcal{O}_{D_X(Y)}/J_Y^{[k]} \otimes (\Omega^i_{Y/T}/\underline{\Omega}^i_{Y/T}).$$

The claim follows from the injectivity on the left. $\square$

The following theorem explains the crystalline property of the filtered complexes we have been considering.

Theorem 6.7. *With the notation of the paragraph above, suppose that $g\colon Y' \to Y$ is a morphism of smooth idealized log schemes over W, and that $i'\colon X \to Y$; and $i := g \circ Y$ are closed immersions, so that g induces morphisms of strict formal completions $\hat{Y}' \to \hat{Y}$ and strict PD-envelopes*

$$D' := D_X(Y') \to D := D_X(Y),$$

1. *The morphism g induces quasi-isomorphisms:*

$$\mathcal{O}_D \otimes \Omega^{\cdot}_{\hat{Y}/W} \to \mathcal{O}_{D'} \otimes \Omega^{\cdot}_{\hat{Y}'/W}$$

$$\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{\hat{Y}/W} \to \mathcal{O}_{D'} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}'/W}$$

2. *The natural maps*

$$\begin{aligned} g^*\colon (\mathcal{O}_D \otimes \Omega^{\cdot}_{\hat{Y}/W}, Fil^{\cdot}_X) &\to (\mathcal{O}_{D'} \otimes \Omega^{\cdot}_{\hat{Y}'/W}, Fil^{\cdot}_X) \\ g^*\colon (\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{\hat{Y}/W}, Fil^{\cdot}_X) &\to (\mathcal{O}_{D'} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}'/W}, Fil^{\cdot}_X) \end{aligned}$$

are filtered quasi-isomorphisms.

Proof. Statement (1) of the theorem is just a special case of statement (2). We begin the proof of statement (2) with the case in which $Y = \mathbf{A}^r_{Y'}$ and $g\colon Y' \to Y$ is the zero section. Let $p\colon Y \to Y'$ be the projection. We shall verify that g induces quasi-isomorphisms between the filtered complexes described in the theorem; since $p \circ g = \mathrm{id}_Y$, it will follow that the same is true for p. An induction argument reduces us to the case in which $r = 1$. Working locally, we assume that $Y' = \operatorname{Spf}((Q, K) \to B)$ and let J' be the ideal of X in Y'.

Then $Y = \mathrm{Spf}(Q, K) \to B\{t\})$, and the ideal of X in Y is (J', t). Let C be the completed PD-envelope J' in B. The PD-envelope of (J', t) in $B\{t\}$ can be identified with the completion of the PD-polynomial algebra $C\langle t\rangle$. Thus every element of $\mathcal{O}_D \otimes \Omega^i_{Y/W}$ can be written uniquely as a formal sum

$$\omega = \sum t^{[j]}(\alpha_j + dt \wedge \beta_j)$$

with $\alpha_j \in \mathcal{O}_{D'} \otimes \Omega^i_{Y'/W}, \beta_j \in \mathcal{O}_{D'} \otimes \Omega^{i-1}_{Y'/W}$, and $\lim \alpha_j = \lim \beta_j = 0$. We have a commutative diagram of filtered complexes:

$$\begin{tikzcd}
(\mathrm{Ker}^{\cdot}, Fil^{\cdot}_X) \arrow[r] & (\mathcal{O}_D \otimes \Omega^{\cdot}_{Y/W}, Fil^{\cdot}_X) \arrow[r] & (\mathcal{O}_{D'} \otimes \Omega^{\cdot}_{Y'/W}, Fil^{\cdot}_X) \\
(\underline{\mathrm{Ker}}^{\cdot}, Fil^{\cdot}_X) \arrow[r] \arrow[u] & (\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{Y/W}, Fil^{\cdot}_X) \arrow[r] \arrow[u] & (\mathcal{O}_{D'} \otimes \underline{\Omega}^{\cdot}_{Y'/W}, Fil^{\cdot}_X) \arrow[u]
\end{tikzcd}$$

The rows are strictly short exact, and Proposition 6.4 shows that the vertical arrows are strict inclusions. The element ω lies in $\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{Y/W}$ if and only if each α_j and β_j do, and ω lies in Ker^i if and only if $\alpha_0 = 0$.

Now define $\rho\colon \mathcal{O}_D \otimes \Omega^i_{Y/W} \to \mathcal{O}_D \otimes \Omega^{i-1}_{Y/W}$ by

$$\sum_{j\geq 0} t^{[j]}(\alpha_j + dt \wedge \beta_j) \mapsto \sum_{j\geq 0} t^{[j+1]}\beta_j,$$

noting that ρ preserves the subcomplexes $\mathrm{Ker}^{\cdot}$ and $\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{Y/W}$ as well as the filtration $Fil^{\cdot}_X$. We calculate:

$$\begin{aligned}
(d\rho + \rho d)(\omega) &= d\left(\sum_{j\geq 0} t^{[j+1]}\beta_j\right) + \\
&\qquad \rho\left(\sum_{j\geq 1}(t^{[j-1]}dt \wedge \alpha_j + \sum_{j\geq 0} t^{[j]}d\alpha_j - \sum_{j\geq 0} t^{[j]}dt \wedge d\beta_j)\right) \\
&= \sum_{j\geq 0}(t^{[j]}dt \wedge \beta_j + t^{[j+1]}d\beta_j) + \sum_{j\geq 1}(t^{[j]}\alpha_j - \sum_{j\geq 0} t^{[j+1]}d\beta_j). \\
&= \omega - \alpha_0
\end{aligned}$$

It follows that $d\rho - \rho d$ is the identity on $\mathrm{Ker}^{\cdot}$ and its filtered subcomplexes and hence that g^* and p^* are indeed filtered quasi-isomorphisms.

The general case follows easily. First note that the map $\hat{g}\colon \hat{Y} \to \hat{Y}'$ is necessarily strict, since $X \to \hat{Y}$ and $X \to \hat{Y}'$ are strict. Suppose that g is smooth. Then $\hat{g}$ is strict and smooth, and hence $\underline{\hat{g}}$ is smooth. Working étale locally, we may suppose that $\hat{g}$ looks like projection from formal affine space over Y, so that the previous argument applies. Next suppose that g is a closed immersion. Then since Y' is smooth and we are working locally, we may assume that g admits a smooth retraction r. Since the result is true for r and for $\mathrm{id}_{Y'}$ and since $r \circ g = \mathrm{id}_{Y'}$, the result also holds for g. For the general case, recall that every morphism can locally factored as a composition of a closed immersion and a smooth morphism. $\square$

In order to relate these crystalline constructions to the de Rham-Witt complex, we begin by supposing that X admits an embedding into a smooth Y/W which is endowed with a Frobenius lifting $\phi_Y \colon Y \to Y$. Then ϕ_Y induces a Frobenius lifting ϕ_D of the strict PD-envelope $D_X(Y)$ of X in Y, which in turn induces an endomorphism $\phi_D^{\cdot}$ of $\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{Y/W}$. Since these complexes are p-torsion free and $\phi_D^{\cdot}$ is visibly divisible by p^i in degree i, these data define a Dieudonné complex $(\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{Y/W}, d, F)$. Arguing as in Steps 1 and 2 of Theorem 5.2, we see that this complex has the structure of a Dieudonné algebra, and that there is a pair of adjoint maps:

$$\mathcal{O}_X \to \mathcal{W}_1 Sat(\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{Y/W})^0, \quad c_Y \colon \mathcal{W}\Omega^{\cdot}_{\underline{X}} \to \mathcal{W}Sat(\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{Y/W}) \tag{6.1}$$

Theorem 6.8. *Let X/k be a fine, saturated, smooth, and reduced idealized log scheme.*

1. *If $i \colon X \to Y$ is an embedding into a fine, saturated and smooth formal idealized log scheme Y over W endowed with a Frobenius lift, then the associated Dieudonné algebra $(\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{\hat{Y}/W}, d, F)$ is quasi-saturated, and the natural map $c_Y \colon \mathcal{W}\Omega^{\cdot}_{\underline{X}} \to \mathcal{W}Sat(\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{Y/W})$ is an isomorphism.*
2. *If Y/W is a smooth formal lifting of X (not necessarily endowed with a Frobenius lift), there is a natural derived isomorphism*

$$(\mathcal{W}\Omega^{\cdot}_{\underline{X}}, d) \sim (\underline{\Omega}^{\cdot}_{\hat{Y}/W}, d).$$

Proof. To prove statement (1), we may work locally on X. Thus we may assume that (X, F_X) admits a smooth formal lifting $(Y', \phi_{Y'})$. By Proposition 1.12, the map $c_{Y'}$ is an isomorphism. Since $(\underline{\Omega}^{\cdot}_{Y'/W}, d, F)$ is of Cartier type and p-adically separated and complete, it is quasi-saturated. Let $(Y'', \phi_{Y''}) := (Y \times Y', \phi_Y \times \phi_{Y'})$. By Theorem 6.7, the map $(Y'', \phi_{Y''}) \to (Y', \phi_{Y'})$ induces a quasi-isomorphism $(\underline{\Omega}^{\cdot}_{\hat{Y}'/W}, d, F) \to (\mathcal{O}_{D''} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}''/W}, d, F)$, and hence by Corollary 1.9, the complex $(\mathcal{O}_{D''} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}''/W}, d, F)$ is also quasi-saturated and the map

$$\mathcal{W}Sat(\mathcal{O}_{D'} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}'/W}, d, F) \to \mathcal{W}Sat(\mathcal{O}_{D''} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}''}, d, F)$$

is an isomorphism. The same corollary applied to the map $(Y'', \phi_{Y''}) \to (Y, \phi_Y)$ shows that $(\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{\hat{Y}/W}, d, F)$ is quasi-saturated and that the map

$$\mathcal{W}Sat(\mathcal{O}_D \otimes \underline{\Omega}^{\cdot}_{\hat{Y}/W}, d, F) \to \mathcal{W}Sat(\mathcal{O}''_D \otimes \underline{\Omega}^{\cdot}_{\hat{Y}''/W}, d, F)$$

is an isomorphism. It follows that c_Y is also an isomorphism.

To deduce statement (2), let $Y^0 \to Y$ be an open affine cover of Y, let $Y^n := Y^0 \times_Y \cdots \times_Y Y^0$ ($n+1$-times), and let X^n be its reduction modulo p. Now let Z^n be the strict formal completion of $Y^0 \times_W \cdots \times_W Y^0$ along X^n and let D^n be the strict PD envelope of X^n in Z^n. [4] We find immersions of simplicial formal idealized log schemes:

$$X^\bullet \to Y^\bullet \to D^\bullet.$$

Since Y^0 is formally smooth and affine, it admits a Frobenius lift ϕ_{Y^0}, which induces Frobenius lifts on $Z^\bullet$ and $D^\bullet$ (but not $Y^\bullet$). We find a diagram:

$$\begin{tikzcd}
C^\cdot(\mathcal{W}\Omega^\cdot_{\underline{X}^\bullet}) \arrow[r, "b"] & C^\cdot(\mathcal{W}Sat(\mathcal{O}_{D^\bullet} \otimes \underline{\Omega}^\cdot_{Z^\bullet/W})) & C^\cdot(\mathcal{O}_{D^\bullet} \otimes \underline{\Omega}^\cdot_{Z^\bullet/W}) \arrow[l, "c"'] \arrow[d, "e"] \\
\mathcal{W}\Omega^\cdot_{\underline{X}} \arrow[u, "a"] & \underline{\Omega}^\cdot_{Y/W} \arrow[r, "f"] & C^\cdot(\underline{\Omega}^\cdot_{Y/W}).
\end{tikzcd}$$

The arrows a and f are quasi-isomorphisms by descent, and arrow b is a quasi-isomorphism by statement (1). Arrow c is a quasi-isomorphism because $\mathcal{O}_D \otimes \underline{\Omega}^\cdot_{Z/W}$ is quasi-saturated, and arrow e is a quasi-isomorphism by Theorem 6.7. A standard simplicial argument, which we will not write out, shows that the resulting derived isomorphism is independent of the choices made and, in fact, is natural. $\square$

Corollary 6.9. *Let Y/W be a fine saturated smooth and proper log scheme over W and let X/k be its reduction modulo p. Choose an embedding $W \to \mathbf{C}$. Then there are natural isomorphisms:*

$$H^\cdot(X, \mathcal{W}\Omega^\cdot_{\underline{X}}) \to H^\cdot(Y, \underline{\Omega}^\cdot_{Y/W}) \otimes_W \mathbf{C} \to H^\cdot(Y_{\mathrm{an}}, \underline{\Omega}^\cdot_{Y/\mathbf{C}}) \leftarrow H^\cdot(Y_{\mathrm{an}}, \mathbf{C})$$

Moreover, the filtration on $H^\cdot(Y, \underline{\Omega}^\cdot_{Y/W})$ coming from the "filtration bête" of $\Omega^\cdot_{Y/W}$ coincides with the Hodge filtration of the mixed Hodge structure on $H^\cdot(Y_{\mathrm{an}}, \mathbf{C})$.

Proof. These results follow from the compatibility of formation of cohomology with flat base change, GAGA, and Danilov's theorems [6, Theorem 3.4]. $\square$

Theorem 6.10. *Let Y/W be a fine saturated smooth and proper idealized log scheme over W and let X/k be its reduction modulo p. Assume the dimension of X/k is less than p.*

1. *There is a derived isomorphism:*

$$(\underline{\Omega}^\cdot_{Y/W}, pd) \sim (\underline{\Omega}^\cdot_{Y/W}, d).$$

[4] We note that Z^n is endowed with the idealized structure coming from any of the maps $p_i \colon Z^n \to Y^0$; these are all the same because, after the exactification used in the construction, $p_i^\flat(m)$ and $p_j^\flat(m)$ differ by a unit.

2. *The Hodge and conjugate spectral sequences of the hypercohomology of the complex $\underline{\Omega}^{\cdot}_{X/k} \cong \mathcal{W}_1\Omega^{\cdot}_{\underline{X}}$ degenerate at E_1 and E_2 respectively.*

Proof. Let us first assume that there exists a locally closed strict embedding $Y \to Y'$, where Y'/W is a fine saturated and smooth idealized log scheme over W which is endowed with a Frobenius lifting $\phi_{Y'}$. Let $D' := D_X(Y')$, and observe that the endomorphism $\phi^*_{Y'}$ of $\mathcal{O}_{D'}$ induced by $\phi_{Y'}$ takes $\overline{J}_{Y'}$ into $p\mathcal{O}_{D'}$, hence $\overline{J}^{[m]}_{Y'}$ into $p^{[m]}\mathcal{O}_{D'}$ for every $m \in \mathbf{N}$. Since $\phi^*_{Y'}$ is also divisible by p^i on $\underline{\Omega}^i_{\hat{Y}'/W}$, we find a morphism of complexes:

$$\Phi_m \colon Fil^m_X(\mathcal{O}_{D'} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}'/W}) \to (p^{[m]})\mathcal{O}_{D'} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}'/W}$$

Lemma 6.11. *With the notations above, suppose that* $\dim X \le m < p$.

1. *Multiplication by p^{m-i} in degree i defines an isomorphism of complexes:*

$$\Psi_m \colon (\underline{\Omega}^{\cdot}_{Y/W}, pd) \to (Fil^m_X\underline{\Omega}^{\cdot}_{Y/W}, d)$$

2. *The morphism Φ_m is a quasi-isomorphism.*

Proof. The ideal of X in Y is just the ideal $p\mathcal{O}_Y$, so

$$Fil^m_X\underline{\Omega}^i_{Y/W} := (p)^{[m-i]}\underline{\Omega}^i_{Y/W} = p^{m-i}\underline{\Omega}^i_{Y/W},$$

since each $m - i < p$. Since $n := \dim X \le m$, multiplication by p^{m-i} in degree i defines an isomorphism of complexes:

$$\begin{array}{ccccccccc}
(\underline{\Omega}^{\cdot}_{Y/W}, pd) & = & \mathcal{O}_Y & \xrightarrow{pd} & \underline{\Omega}^1_{Y/W} & \xrightarrow{pd} & \cdots & \underline{\Omega}^n_{Y/W} \\
\downarrow{\scriptstyle \Psi_m} & & \downarrow{\scriptstyle p^m} & & \downarrow{\scriptstyle p^{m-1}} & & \cdots & \downarrow{\scriptstyle p^{m-n}} \\
(Fil^m_X\underline{\Omega}^{\cdot}_{Y/W}, d) & = & p^m\mathcal{O}_Y & \xrightarrow{d} & p^{m-1}\underline{\Omega}^1_{Y/W} & \xrightarrow{d} & \cdots & p^{m-n}\underline{\Omega}^n_{Y/W}
\end{array}$$

This proves statement (1).

Statement (2) can be checked locally, so we may assume that Y is affine and endowed with a Frobenius lift ϕ_Y. Let $\hat{Y}''$ be the strict formal completion of X in $Y' \times Y$, with the Frobenius lift $\phi_{Y''}$ induced by $\phi_{Y'} \times \phi_Y$. Since $D_X(Y) = Y$, we have a commutative diagram:

$$\begin{array}{ccc}
\Phi'_m \colon Fil^m_X(\mathcal{O}_{D'} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}'/W}) & \longrightarrow & (p^{[m]})\mathcal{O}_{D'} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}'/W} \\
\downarrow & & \downarrow \\
\Phi''_m \colon Fil^m_X(\mathcal{O}_{D''} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}''/W}) & \longrightarrow & (p^{[m]})\mathcal{O}_{D''} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}''/W} \\
\uparrow & & \uparrow \\
\Phi_m \colon Fil^m_X(\underline{\Omega}^{\cdot}_{Y/W}) & \longrightarrow & (p^{[m]})\underline{\Omega}^{\cdot}_{\hat{Y}'/W}.
\end{array}$$

The vertical arrows are quasi-isomorphisms by Theorem 6.7 so it will suffice to prove that Φ_m is a quasi-isomorphism. Composing Φ_m with multiplication by p^{-m}, we find the morphism

$$F\colon (\underline{\Omega}^{\cdot}_{Y/W}, pd) \to \underline{\Omega}^{\cdot}_{Y/W}$$

associated to the Dieudonné complex $(\underline{\Omega}^{\cdot}_{Y/W}, d, F)$, which we encountered earlier (1.1). Since this complex is of Cartier type, the reduction of F modulo p is a quasi-isomorphism,. Since the complex is p-torsion free and p-adically separated and complete, it follows that F is also a quasi-isomorphism, and then so is Φ_m. □

To prove statement (1) of the theorem, observe that, if there exists an embedding of Y into a Y' admitting a Frobenius lift, we have a diagram

$$\begin{tikzcd}
 & (Fil^m_X(\mathcal{O}_{D'} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}'/W}, d)) \arrow[r, "\Phi_m"] \arrow[d, "a"'] & (p^m\mathcal{O}_{D'} \otimes \underline{\Omega}^{\cdot}_{\hat{Y}'/W}, d) \arrow[d, "b"] \\
(\underline{\Omega}^{\cdot}_{Y/W}, pd) \arrow[r, "\Psi_m"] & (Fil^m_X\underline{\Omega}^{\cdot}_{Y/W}, d) & (p^m\underline{\Omega}^{\cdot}_{Y/W}, d)
\end{tikzcd}$$

The morphisms Φ_m and Ψ_m are quasi-isomorphisms by Lemma 6.11, and the morphisms a and b are quasi-isomorphisms by Theorem 6.7. Inverting the morphism a in the derived category and composing with the other morphisms gives the desired derived isomorphism, when $Y \to Y'$ exists. A standard simplicial argument will cover the general case.

The reduction modulo p of the derived isomorphism of statement (1) gives a quasi-isomorphism

$$(\underline{\Omega}^{\cdot}_{X/k}, 0) \sim (\underline{\Omega}^{\cdot}_{X/k}, d).$$

In other words, the complex $(\underline{\Omega}^{\cdot}_{X/k}, d)$ is "completely decomposed" in the sense of [9]. The argument there, using a dimension count and the Cartier isomorphism, then applies to prove the theorem. □

7 The Hodge and Nygaard filtrations

Our aim here is to give a brief account of some of the essential features of the construction of the Nygaard filtration as discussed in [4]. We also explain its application to the proof of Katz's conjecture, following Nygaard's method in [19], but adapted to the language of [4].

We begin with a general construction, going back to Mazur's original article [17]. Let p be a fixed natural number, typically a prime. By a *p-span* in an abelian category we mean a monomorphism $\Phi\colon M' \to M$ of p-torsion free objects. A p-span is a *p-isogeny* if there exist a natural number ℓ and a morphism $\Psi\colon M \to M'$ such that $\Phi \circ \Psi = p^\ell \mathrm{id}_M$ and $\Psi \circ \Phi = p^\ell \mathrm{id}_{M'}$. The smallest such ℓ is called the *level* of the isogeny.

Definition 7.1. *If $\Phi\colon M' \to M$ is a p-span, let $\overline{M} := M/pM$, and define, for $i \geq 0$,*

$$\begin{array}{rcl} N^i M' & := & \Phi^{-1}(p^i M) \\ N_i M & := & Im(p^{-i}\Phi\colon N^i M' \to M) \\ N^i \overline{M}' & := & Im(N^i M' \to M'/pM') \\ N_i \overline{M} & := & Im(N_i M \to M/pM) \end{array}$$

The verification of the following proposition is immediate.

Proposition 7.2. *With the definitions above, $N^{\cdot}$ is a descending filtration of M', and $N.$ is an ascending filtration of M. Furthermore*

$$\begin{array}{rcl} pN^{i-1}M' & = & N^i M' \cap pM' \\ pN_{i+1}M & = & N_i M \cap pM, \end{array}$$

The map $p^{-i}\Phi$ induces isomorphisms of pairs

$$\begin{array}{rcl} (N^i M', N^{i+1}M') & \xrightarrow{\cong} & (N_i M, pN_{i+1}M) \\ (N^i M', pN^{i-1}M') & \xrightarrow{\cong} & (N_i M, N_{i-1}M) \\ (N^i M', N^{i+1}M' + pN^{i-1}) & \xrightarrow{\cong} & (N_i M, N_{i-1}M + pN_{i+1}M) \end{array}$$

and hence isomorphisms:

$$\begin{array}{rcl} \mathrm{Gr}^i_N M' & \xrightarrow{\cong} & N_i \overline{M} \\ N^i \overline{M}' & \xrightarrow{\cong} & \mathrm{Gr}^N_i M \\ \mathrm{Gr}^i_N \overline{M}' & \xrightarrow{\cong} & \mathrm{Gr}^N_i \overline{M}. \end{array}$$

□

It follows from the definitions that $N^0\overline{M}' = \overline{M}'$ and that $N_{-1}\overline{M} = 0$. If Φ is a p-isogeny of level ℓ, then $N^{\ell+1}\overline{M}' = 0$ and $N_\ell \overline{M} = \overline{M}$. Formation of these filtrations is natural: a morphism of p-spans induces morphisms of filtered objects in the obvious way.

Example 7.3. Mazur's proof of the Katz conjectures in [17] is based on an analysis of p-isogenies in the category of of finitely generated W-modules. For example, let i be a natural number and let $\Phi\colon M' \to M$ denote multiplication by p^i on W. Then $N^{\cdot}\overline{M}'$ (resp. $N.\overline{M}$) is the unique filtration on k such that $\mathrm{Gr}^i k$ (resp. $\mathrm{Gr}_i k$) is nonzero. It is a standard fact that every p-isogeny in the category of finitely generated W-modules is a direct sum of objects of this type, and consequently is determined up to isomorphism by its "abstract Hodge numbers" $h^i(\Phi) := \dim_k \mathrm{Gr}^i_N \overline{M}' = \dim_k \mathrm{Gr}^N_i \overline{M}$.

Let $(M^{\cdot}, d, F)$ be a p-torsion free Dieudonné complex with $M^i = 0$ for $i < 0$. Then the morphism $\Phi\colon (M^{\cdot}, d, F) \to (M^{\cdot}, d, F)$, given by $p^i F$ in degree i, defines a p-span in the category of Dieudonné complexes, and hence filtrations $N^{\cdot}$ and $N.$ of $(M^{\cdot}, d, F)$,

Proposition 7.4. *Let $(M^\cdot, d, F)$ be a p-torsion free Dieudonné complex such that $M^n = 0$ for $n < 0$, and let $N^\cdot$ and $N.$ be the filtrations on $M^\cdot$ defined by Φ as in Definition 7.1.*

1. *If $(M^\cdot, d, F)$ is of Cartier type, then*

$$\begin{array}{lllllllll} N^i M^\cdot = p^i M^0 & \to p^{i-1}M^1 & \to \cdots \to & pM^{i-1} & \to M^i & \to M^{i+1} \cdots \\ N^i \overline{M}^\cdot = \quad 0 & \to \quad 0 & \to \cdots \to & \quad 0 & \to \overline{M}^i & \to \overline{M}^{i+1} \cdots \end{array}$$

2. *If $(M^\cdot, d.F)$ is saturated, then*

$$\begin{array}{llllll} N^i M^\cdot = p^{i-1}VM^0 \to & p^{i-2}VM^1 \to & \cdots \to VM^{i-1} & \to M^i & \to M^{i+1} \cdots . \\ N_i M^\cdot = \quad M^0 \quad \to & \quad M^1 \to & \cdots \to \quad M^{i-1} & \to \quad FM^i & \to pFM^{i+1} \to \cdots . \end{array}$$

Furthermore, the inverse of the isomorphism $p^{-i}\Phi \colon N^i M^\cdot \to N_i M^\cdot$ is given by $p^{i-n-1}V$ in degree n. If $M^n = 0$ for $n \geq \ell$, then Φ is a p-isogeny of level at most ℓ.

Proof. An element x of M^n lies in $N^i M^n$ if and only if $p^n F x = p^i y$ for some $y \in M^n$. If $i \leq n$ this condition is vacuous, so $N^i M^n = M^n$ when $i \leq n$. Suppose that $x \in N^i M^n$, that $n < i$, and that $M^\cdot$ is of Cartier type. It follows that $Fx \in pM^n$, hence x is killed by the isomorphism $\gamma \colon M^n/pM^n \to H^n(M^\cdot/pM^\cdot)$, hence $x \in pM$. Repeating the argument with $p^{-1}x$, we eventually see that $x \in p^{i-n}M^n$. The reverse inclusion is trivial.

Now suppose that $M^\cdot$ is saturated. If $n < i$, we see that $x \in N^i M^n$ if and only if $\Phi(x) \in p^i M^n$, *i.e.*, if and only if $Fx = p^{i-n}y$ for some $y \in M^n$. Then $Fx = p^{i-n-1}py = p^{i-n-1}FVy$, that is, if and only if $x = p^{i-n-1}Vy$. Then $p^{-i}\Phi x = p^{-i}\Phi p^{i-n-1}Vy = p^{n-i}Fp^{i-n-1}Vy = y$, so $N_i M^n = M^n$ when $n < i$. On the other hand, if $i \leq n$, then $N_i M^n := p^{-i}\Phi N^i M^n = p^{n-i}FM^n$. If also $M^n = 0$ for $n > \ell$, then $N_\ell M^\cdot = M^\cdot$ and Φ is a p-isogeny of level $\leq \ell$. $\square$

If $(M^\cdot, d, F)$ is a saturated Dieudonné complex and $(\mathcal{W}M^\cdot, d, F)$ is its completion, we find a natural map of filtered complexes:

$$(M^\cdot, d, N^\cdot) \to (\mathcal{W}M^\cdot, d, N^\cdot).$$

I do not know if this map is strictly compatible with the filtrations. However it is easy to see that, for every $r > 0$, the filtrations $N^\cdot M^\cdot$ and $N^\cdot \mathcal{W}M^\cdot$ induce the same filtration of $\mathcal{W}_r M^\cdot$. Indeed, if $x \in p^j V \mathcal{W}M^n$, then we can find sequences $(y_m) \in M^n, (z_m) \in M^{n-1}$ so that

$$\begin{aligned} x &= p^j V \sum_{m=0}^{\infty} (V^m y_m + dV^m z_m) \\ &= p^j V \sum_{m=0}^{r-1} (V^m y_m + dV^m z_m) + \sum_{m=r}^{\infty} (V^m p^j V y_m + dV^m p^{j+1} V z_m), \end{aligned}$$

so the element $p^jV\sum_{m=0}^{r-1}(V^m y_m + dV^m z_m)$ of p^jVM^n has the same image in $\mathcal{W}_r M^n$ as does x. The analogous statement is true for $N.$: if $x \in p^jF\mathcal{W}M^n$, then

$$\begin{aligned} x &= p^jF\sum_{m=0}^{r}(V^m y_m + dV^m z_m) + p^jF\sum_{m=r+1}^{\infty}(V^m y_m + dV^m z_m) \\ &= p^jF\sum_{m=0}^{r}(V^m y_m + dV^m z_m) + \sum_{m=r+1}^{\infty}(V^m p^jF y_m + dV^{m-1}p^j z_m) \end{aligned}$$

The following result is the promised filtered version of statement (2) of Proposition 1.7 and [4, 2.7.3]. It is related to Nygaard's [19, Theorem 1.5], which is essentially this result except applied to the rth powers of Φ and p and the corresponding filtrations.

Theorem 7.5. *If $(M^\cdot, d, F)$ is a saturated Dieudonné complex, then for every $r > 0$, the natural maps*

$$\pi_r\colon (M^\cdot/p^rM^\cdot, N^\cdot) \to (\mathcal{W}_r M^\cdot, N^\cdot)$$

$$\pi_r\colon (M^\cdot/p^rM^\cdot, N_\cdot) \to (\mathcal{W}_r M^\cdot, N_\cdot)$$

are filtered quasi-isomorphisms.

Proof. Let us write $M_r^\cdot$ for $M^\cdot/p^rM^\cdot$ and $K_r^\cdot$ for the kernel of π_r, *i.e.*, $K_r^n = Fil^rM^n/p^rM^n$. By definition, $N^iM_r^n$ is the image of the map $N^iM^n \to M_r^n$ and $N^iK_r^n := K_r^n \cap N^iM_r^n$, so we have an exact sequence of complexes:

$$0 \to N^iK_r^\cdot \to N^iM_r^\cdot \to N^i\mathcal{W}_rM^\cdot \to 0.$$

There is an analogous sequence with N_i in place of N^i. It will suffice to show that the complexes $N^iK_r^\cdot$ and $N_iK_r^\cdot$ are acyclic.

Let us first check that $K_r^\cdot$ is acyclic. An element of K_r^n is the image of an element x of Fil^rM^n, say $x = V^rx' + dV^rx''$, so $dx = dV^rx'$. If x lifts a cycle, then $dx = p^rz$ for some z. Then $dx' = F^rdV^rx' = F^rdx = F^rp^rz = p^rF^rz$. Since $M^\cdot$ is saturated, it follows that $x' = F^rx'''$ for some x'''. Then $V^rx' = p^rx'''$, so in fact $x \equiv dV^rx'' \pmod{p^rM^n}$. Since $V^rx'' \in Fil^rM^{n-1}$, we see that K_r^n is indeed acyclic.

To see that $N^iK_r^\cdot$ is acyclic, we must show that if x as above belongs to $N^iM + p^rM^n$, then $V^rx'' \in N^iM^{n-1} + p^rM^{n-1}$. If $r = 0$ or $i < n$, there is nothing to check. If $r > 0$ and $i = n$, then $N^iM^{n-1} = VM^{n-1}$ which contains V^rx'' since $r \geq 1$. Suppose $r > 0$ and $i = n + j$ with $j > 0$. Since $x \in N^iM^n + p^rM^n$, we can write $x = p^{j-1}Vz + p^rz'$, and since $x \equiv dV^rx''$ $\pmod{p^rM^n}$, we find that

$$dx'' = F^rdV^rx'' \equiv F^rx \equiv F^rp^{j-1}Vz \equiv p^jF^{r-1}z \pmod{p^rM^n}.$$

If $j \geq r$, then $x \in p^rM^n$ and there is nothing to prove, so we may assume that $j < r$. Then $dx'' \in p^jM^n$, and since M is saturated, we can write $x'' = F^jx'''$, and then

$$V^rx'' = V^{r-j}V^jF^jx''' = p^jVV^{r-j-1}x''' \in N^iM^{n-1},$$

as required.

The proof of the second part is similar. If $x \in M^n$ lifts a cycle of $N_i K_r^n$, then as before we can write $x = dV^r x'' + p^r z''$ and $x = p^j F z + p^r z'$; without loss of generality $j < r$. Then $dx'' = F^r dV^r x'' \equiv F^r x \equiv p^j F^{r+1} z$, so there exists x''' such that $x'' = F^j x'''$. Then $V^r x'' = p^{j-1} F V^{r-j+1} x''' \in N_i M^{n-1}$. $\square$

The second part of the following result is contained in [4, 8.2.1].

Corollary 7.6. *If $(M^\cdot, d, F)$ is a saturated Dieudonné complex, there are natural quasi-isomorphisms:*

$$\begin{array}{ccccccccc}
N^i \overline{M}^\cdot & = & \cdots 0 & \longrightarrow & VM^{i-1}/pM^{i-1} & \longrightarrow & \overline{M}^i & \longrightarrow & \overline{M}^{i+1} \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\beta^{\geq i} \mathcal{W}_1 M^\cdot & = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{W}_1 M^i & \longrightarrow & \mathcal{W}_1 M^{i+1} \cdots
\end{array}$$

and

$$\begin{array}{ccccccccc}
N_i \overline{M}^\cdot & = & \overline{M}^0 & \longrightarrow & \cdots \quad \overline{M}^{i-1} & \longrightarrow & FM^i/pM^i & \longrightarrow & 0 \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\tau^{\leq i} \overline{\mathcal{W}}_1 M^\cdot & = & \mathcal{W}_1 M^0 & \longrightarrow & \cdots \quad \mathcal{W}_1 M^{i-1} & \longrightarrow & Z^i(\mathcal{W}_1 M^\cdot) & \longrightarrow & 0 \cdots
\end{array}$$

Proof. The first statement is just the special case of Theorem 7.5 when $r = 1$. Since $M^\cdot$ is saturated, we can identify FM^i/pM^i with $Z^i(\overline{M}^\cdot)$, and so $N_i \overline{M}^\cdot$ identifies with $\tau^{\leq i} \overline{M}^\cdot$. Since $\overline{M}^\cdot \to \mathcal{W}_1 M^\cdot$ is a quasi-isomorphism, the same holds after applying $\tau^{\leq i}$, and the result follows. $\square$

The next result is an easy consequence of the previous one, but it is just as easy to check it directly.

Corollary 7.7. *If $(M^\cdot, d, F)$ is a saturated Dieudonné complex, there is a commutative diagram of quasi-isomorphisms:*

$$\begin{array}{ccc}
\mathrm{Gr}_N^i \overline{M}^\cdot & \longrightarrow & \mathcal{W}_1 M^i[-i] \\
\downarrow & & \downarrow \\
\mathrm{Gr}_i^N \overline{M}^\cdot & \longrightarrow & H^i(\mathcal{W}_1 M^\cdot)[-i],
\end{array}$$

where the horizontal arrows are induced by the arrows of Theorem 7.5, the left vertical arrow is the one appearing in the last line of Proposition 7.2 and the right vertical arrow is the Cartier isomorphism ψ_1 of Proposition 1.7. $\square$

The following result shows that the map from a Dieudonné complex of Cartier type to its saturation is a filtered quasi-isomorphism, mod powers of p. (Compare with [4, 8.3.4 and 8.3.5].)

Proposition 7.8. *If $(M^\cdot, d, F)$ is a p-torsion free Dieudonné complex and $r \in \mathbf{N}$, let $M^\cdot_r := M^\cdot/p^r M^\cdot$ and let $N^i M^\cdot_r$ denote the image of $N^i M^\cdot$ in $M^\cdot_r$. Then if $(M^\cdot, d, F)$ is of Cartier type and $M^n = 0$ for $n < 0$, then for all i and all r, the natural maps:*

$$
\begin{array}{lrcl}
(7.8.1) & \mathrm{Gr}^i_N \overline{M}^\cdot & \to & \mathrm{Gr}^i_N \overline{Sat(M}^\cdot) \\
(7.8.2) & N^i \overline{M}^\cdot & \to & N^i \overline{Sat(M}^\cdot) \\
(7.8.3) & N^i M^\cdot_r & \to & N^i \,\mathrm{Sat}(M^\cdot_r) \\
(7.8.4) & M^\cdot/N^i M^\cdot & \to & \mathrm{Sat}\, M^\cdot/N^i(Sat(M^\cdot))
\end{array}
$$

are quasi-isomorphisms.

Proof. Statement (1) of Proposition 7.4 shows that $\mathrm{Gr}^i_N \overline{M}$ is just $\overline{M}^i[-i]$. Composing the map (7.8.1) with the quasi-isomorphism $\mathrm{Gr}^i_N \overline{SatM}^\cdot \to \mathcal{W}_1 SatM^i[-i]$ of Corollary 7.7, we find a map $\overline{M}^i[-i] \to \mathcal{W}_1 SatM^i[-i]$, which is nothing but the isomorphism in the last statement of Theorem 1.8. It follows that the map (7.8.1) is a also a quasi-isomorphism, and then induction shows that the same is true of (7.8.2). Since $N^i M^n \cap p^r M^n = p^r N^{i-r} M^n$ (and similarly for $SatM$), we have a commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N^{i-r}\overline{M}^\cdot & \longrightarrow & N^i M^\cdot_{r+1} & \longrightarrow & N^i M^\cdot_r & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N^{i-r}\overline{(}SatM^\cdot) & \longrightarrow & N^i Sat(M^\cdot)_{r+1} & \longrightarrow & N^i Sat(M)^\cdot_r & \longrightarrow & 0
\end{array}
$$

Then another induction proves that (7.8.3) is also a quasi-isomorphism. Since $p^r M^n \subseteq N^i M^n$ for $r \gg 0$, (and similarly for $\mathrm{Sat}(M)$) it follows that (7.8.4) is a quasi-isomorphism as well. $\square$

The following result shows that, under suitable hypotheses, formation of the filtrations $N^\cdot$ and $N.$ commutes with passage to hypercohomology.

Proposition 7.9. *Let $(M^\cdot, F, d)$ be a strict Dieudonné complex on a topological space (or topos) X. Suppose that the following hypotheses are satisfied.*

1. *The groups $H^\cdot(X, M^\cdot)$ are p-torsion free.*
2. *The two spectral sequences of hypercohomology associated to the complex $\mathcal{W}_1 M^\cdot$ degenerate, at E_1 and at E_2 respectively. That is:*
   - (a) *For all i, the maps $H^\cdot(X, \beta^{\geq i}\mathcal{W}_1 M^\cdot) \to H^\cdot(X, \mathcal{W}_1 M^\cdot)$ are injective.*
   - (b) *For all i, the maps $H^\cdot(X, \tau^{\leq i}\mathcal{W}_1 M^\cdot) \to H^\cdot(X, \mathcal{W}_1 M^\cdot)$ are injective.*

Let $N^{\cdot}$ and $N.$ be the filtrations on $H^{\cdot}(X, M^{\cdot})$ defined by the map

$$H^{\cdot}(\Phi)\colon H^{\cdot}(X, M^{\cdot}) \to H^{\cdot}(X, M^{\cdot})$$

as in Definition 7.1. Then the following conclusions hold.

1. *For all i, the natural maps*

$$H^{\cdot}(X, M^{\cdot})/p^i H^{\cdot}(X, M^{\cdot}) \to H^{\cdot}(X, M^{\cdot}/p^i M^{\cdot})$$

are isomorphisms. In particular, the natural maps

$$H^{\cdot}(X, M^{\cdot})/pH^{\cdot}(X, M^{\cdot}) \to H^{\cdot}(X, \overline{M}^{\cdot}) \to H(X, \mathcal{W}_1 M^{\cdot})$$

are isomorphisms.

2. *The natural maps*

$$\begin{array}{ccc} H^{\cdot}(X, N^i M^{\cdot}) & \longrightarrow & N^i H^{\cdot}(X, M^{\cdot}) \\ H^{\cdot}(X, N_i M^{\cdot}) & \longrightarrow & N_i H^{\cdot}(X, M^{\cdot}) \end{array}$$

are isomorphisms.

3. *The natural maps*

$$\begin{array}{ccc} H^{\cdot}(X, N^i M^{\cdot}) & \longrightarrow & H^{\cdot}(X, \beta^{\geq i}\mathcal{W}_1 M^{\cdot}) \\ H^{\cdot}(X, N_i M^{\cdot}) & \longrightarrow & H^{\cdot}(X, \tau^{\leq i}\mathcal{W}_1 M^{\cdot}) \end{array}$$

are surjective.

Proof. Conclusion (1) follows from the long exact cohomology sequence associated to the short exact sequence

$$0 \to M^{\cdot} \xrightarrow{p^i} M^{\cdot} \to M^{\cdot}/p^i M^{\cdot} \to 0,$$

hypothesis (1), and the fact that $\overline{M}^{\cdot} \to \mathcal{W}_1 M^{\cdot}$ is a quasi-isomorphism (see Proposition 1.7).

The proof of the following lemma depends on the degeneration of the first hypercohomology spectral sequence.

Lemma 7.10. *For every i, the map $H^{\cdot}(X, N^i M^{\cdot}) \to H^{\cdot}(X, M^{\cdot})$ is injective.*

Proof. We use induction on i, the case $i = 0$ being trivial. Thanks to Proposition 7.2, we have an exact sequence

$$0 \to N^{i-1}M^{\cdot} \xrightarrow{[p]} N^i M^{\cdot} \to N^i \overline{M}^{\cdot} \to 0 \tag{7.1}$$

and hence a commutative diagram in which the rows are exact:

$$\begin{tikzcd} H^{\cdot}(X, N^{i-1}M^{\cdot}) \arrow[r, "{[p]}"] \arrow[d, "a_{i-1}"'] & H^{\cdot}(X, N^i M^{\cdot}) \arrow[r] \arrow[d, "a_i"] & H^{\cdot}(X, N^i \overline{M}^{\cdot}) \arrow[d, "b_i"] \\ H^{\cdot}(X, M^{\cdot}) \arrow[r, "p"] & H^{\cdot}(X, M^{\cdot}) \arrow[r] & H^{\cdot}(X, \overline{M}^{\cdot}). \end{tikzcd}$$

The map a_{i-1} is injective by the induction hypothesis, the map p in the lower left is injective because $H^{\cdot}(M)$ is torsion free, and by Theorem 7.5 the map b_i identifies with the map $H^{\cdot}(\beta^{\geq i}\mathcal{W}_1 M^{\cdot}) \to H^{\cdot}(\mathcal{W}_1 M^{\cdot})$, which is injective by hypothesis (2a). It follows that a_i is injective. $\square$

Since $N^i M^{\cdot}$ is the kernel of the map

$$M^{\cdot} \xrightarrow{\Phi} M^{\cdot} \to M^{\cdot}/p^i M^{\cdot},$$

we find a map

$$\phi_i \colon M^{\cdot}/N^i M^{\cdot} \to M^{\cdot}/p^i M^{\cdot}$$

The next lemma uses the hypothesis that the second hypercohomology sequence degenerates.

Lemma 7.11. *For every i, the map $H^{\cdot}(X, M^{\cdot}/N^i M^{\cdot}) \to H^{\cdot}(X, M^{\cdot}/p^i M^{\cdot})$ induced by ϕ_i is injective.*

Proof. We argue by induction on i, the case $i = 0$ being trivial. Let ρ_i be the composition

$$\rho_i : \mathrm{Gr}^i_N M^{\cdot} \xrightarrow{\alpha_i} N_i \overline{M}^{\cdot} \to \overline{M}^{\cdot},$$

where the first arrow is the isomorphism from Proposition 7.2 and the second is the evident inclusion. We have a commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Gr}^i_N M^{\cdot} \arrow[r] \arrow[d, "\rho_i"'] & M^{\cdot}/N^{i+1} M^{\cdot} \arrow[r] \arrow[d, "\phi_{i+1}"'] & M^{\cdot}/N^i \arrow[r] \arrow[d, "\phi_i"] & 0 \\
0 \arrow[r] & M^{\cdot}/pM^{\cdot} \arrow[r, "{[p^i]}"] & M^{\cdot}/p^{i+1} M^{\cdot} \arrow[r] & M^{\cdot}/p^i M^{\cdot} \arrow[r] & 0
\end{tikzcd}$$

with exact rows. This yields the diagram:

$$\begin{tikzcd}
H^{\cdot}(X, \mathrm{Gr}^i_N M^{\cdot}) \arrow[r] \arrow[d, "H^{\cdot}(\rho_i)"'] & H^{\cdot}(X, M^{\cdot}/N^{i+1} M^{\cdot}) \arrow[r] \arrow[d, "H^{\cdot}(\phi_{i+1})"'] & H^{\cdot}(X, M^{\cdot}/N^i M^{\cdot}) \arrow[d, "H^{\cdot}(\phi_i)"] \\
H^{\cdot}(X, \overline{M}^{\cdot}) \arrow[r, "{[p^i]}"] & H^{\cdot}(X, M^{\cdot}/p^{i+1} M^{\cdot}) \arrow[r] & H^{\cdot}(X, M^{\cdot}/p^i M^{\cdot}).
\end{tikzcd}$$

The rows in the diagram are exact, the map labeled $[p^i]$ is injective by hypothesis (1), and the map $H^{\cdot}(\phi_i)$ is injective by the induction hypothesis. The map $H^{\cdot}(\rho_i)$ factors as a composite

$$H^{\cdot}(X, \mathrm{Gr}^i_N M^{\cdot}) \xrightarrow{H^{\cdot}(\alpha_i)} H^{\cdot}(X, N_i \overline{M}^{\cdot}) \xrightarrow{\beta_i} H^{\cdot}(\overline{M}^{\cdot});$$

The first map is an isomorphism since α_i is, and by Theorem 7.5, the map β_i identifies with the map $H^{\cdot}(X, \tau^{\leq i}\mathcal{W}_1 M^{\cdot}) \to H^{\cdot}(X, \mathcal{W}_1 M^{\cdot})$, which is injective by hypothesis (2b). It follows that $H^{\cdot}(\rho_i)$ is injective and then that $H^{\cdot}(\phi_{i+1})$ is injective.

$\square$

Lemma 7.12. *The map $H^\cdot(X, N^i M^\cdot) \to H^\cdot(X, N^i \overline{M}^\cdot)$ is surjective.*

Proof. The exact sequence (7.1) yields a long exact sequence

$$H^\cdot(X, N^i M^\cdot) \to H^\cdot(X, N^i \overline{M}^\cdot) \to H^{\cdot+1}(X, N^{i-1} M^\cdot) \xrightarrow{[p]} H^{\cdot+1}(X, N^i M^\cdot).$$

Thus it suffices to show that the map $[p]$ is injective. This follows from the commutative diagram

$$\begin{tikzcd}
H^\cdot(X, N^{i-1} M^\cdot) \arrow[r, "{[p]}"] \arrow[d] & H^\cdot(X, N^i M^\cdot) \arrow[d] \\
H^\cdot(X, M^\cdot) \arrow[r, "p"] & H^\cdot(X, M^\cdot),
\end{tikzcd}$$

the torsion freeness of $H^\cdot(X, M^\cdot)$, and Lemma 7.10. $\square$

Now to prove the theorem, recall that $N^i H^\cdot$ is by definition the kernel of the composition

$$c_i \colon H^\cdot(X, M^\cdot) \xrightarrow{H^\cdot(\Phi)} H^\cdot(X, M^\cdot) \longrightarrow H^\cdot(X, M^\cdot)/p^i H^\cdot(X, M^\cdot).$$

The top row of the following commutative diagram is exact:

$$\begin{tikzcd}
H^\cdot(X, N^i M^\cdot) \arrow[r, "a_i"] & H^\cdot(X, M^\cdot) \arrow[r] \arrow[d, "c_i"'] & H^\cdot(X, M^\cdot/N^i M^\cdot) \arrow[d, "\phi_i"] \\
 & H^\cdot(X, M^\cdot)/p^i H^\cdot(X, M^\cdot) \arrow[r, "\cong"] & H^\cdot(X, M^\cdot/p^i M^\cdot).
\end{tikzcd}$$

As we have seen, a_i and ϕ_i are injective, and it follows that $H^\cdot(X, N^i M^\cdot)$ identifies with the kernel of c_i. $\square$

Let us sketch how Proposition 7.9 implies Katz's conjecture for smooth proper log schemes, generalizing Mazur's classic theorem [17]. Note that, thanks to Theorem 6.10, the hypothesis of Hodge degeneration is automatically satisfied if the dimension of X is less than p and if it admits a log structure such that the associated log scheme lifts smoothly to W.

Theorem 7.13. *Let X/k be a smooth proper ideally toroidal scheme over a perfect field k of characteristic $p > 0$. and let $H^\cdot_{dRW}(X) := H^\cdot(X, \mathcal{W}\Omega^\cdot_{\underline{X}})$. Assume that $H^\cdot_{dRW}(X)$ is torsion free and that the Hodge and/or conjugate spectral sequence of $\mathcal{W}_1\Omega^\cdot_X$ degenerates at E_1. Let Φ denote the endomorphism of $H^\cdot_{dRW}(X)$ induced by F_X and let $N^\cdot$ and $N_\cdot$ be the corresponding filtrations of $H^\cdot_{dRW}(X)$ as in Definition 7.1.*

1. *The natural map* $\overline{H}^{\cdot} := H^{\cdot}_{dRW}(X)/pH^{\cdot}_{dRW}(X) \to H^{\cdot}(X, \mathcal{W}_1\Omega^{\cdot}_X)$ *is an isomorphism.*

2. *The filtration induced by* $N^{\cdot}$ *on* $H(X, \mathcal{W}_1\Omega^{\cdot}_X)$ *is the Hodge filtration.*

3. *The filtration induced by* $N.$ *on* $H(X, \mathcal{W}_1\Omega^{\cdot}_X)$ *is the conjugate filtration.*

4. *The dimension of* $\mathrm{Gr}^i_N \overline{H}^n$ *is equal to the dimension of* $H^{n-i}(X, \mathcal{W}_1\Omega^i_{X/k})$.

5. *The Newton polygon of the action of* Φ *on* $H^{\cdot}_{dRW}(X)$ *lies on or above the Hodge polygon of* X/k *in degree* n.

Proof. We know from Proposition 4.3 that the cohomology groups $H^q(X, \mathcal{W}_1\Omega^i_X)$ are finite dimensional. The Cartier isomorphism (see (3) of Proposition 1.7) implies that $H^q(X, \mathcal{W}_1\Omega^i_X) \cong H^q(X, \mathcal{H}^i(\mathcal{W}_1\Omega^{\cdot}_X))$. Thus the dimensions of the E_1 terms of the "Hodge" spectral sequence match the dimensions of the E_2 terms of the "conjugate" spectral sequence, so if one of these degenerates, so does the other. Then Statements (1)–(4) follow from Proposition 7.9. Statement (5) follows, since the Newton polygon of an F-crystal always lies on or above the polygon formed from the numbers $\dim \mathrm{Gr}^i_N \overline{H}$ [17]. □

Finally, we give a give a relatively computation free proof of the theorem of Langer-Zink [16, 4.7], comparing the Nygaard filtration of the de Rham-Witt complex with the Hodge filtration on crystalline cohomology, generalized here to the logarithmic case.

Theorem 7.14. *Let* X/k *be a fine saturated and smooth idealized log scheme, strictly embedded in a smooth formal* Y/W. *Then if* $i < p$, *there is a natural derived isomorphism*

$$Fil^i_X(\mathcal{O}_D \otimes \Omega^{\cdot}_{Y/W}) \sim N^i\mathcal{W}\Omega^{\cdot}_{\underline{X}}.$$

Proof. Using the standard simplicial argument, we reduce to the case in which Y is endowed with a Frobenius lift Φ_Y. Then $\mathcal{O}_D \otimes \Omega^{\cdot}_{Y/W}$ has the structure of a Dieudonné algebra, and as we saw in Theorem 6.8, $\mathcal{W}\Omega^{\cdot}_X$ can be identified with its completed saturation. Thus there is a natural map of Dieudonné algebras :

$$c : \mathcal{O}_D \otimes \Omega^{\cdot}_{Y/W} \to \mathcal{W}\Omega^{\cdot}_{\underline{X}}.$$

The endomorphism Φ^*_Y of $\mathcal{O}_D \otimes \Omega^{\cdot}_{Y/W}$ induced by Φ_Y is divisible by p^i on $Fil^i_X(\mathcal{O}_D \otimes \Omega^{\cdot}_{Y/W})$ and hence c induces a morphism

$$c^i_Y : Fil^i_X(\mathcal{O}_D \otimes \Omega^{\cdot}_{\tilde{Y}/W} \to N^i\mathcal{W}\Omega^{\cdot}_{\underline{X}} := {\Phi^*_Y}^{-1}(p^i\mathcal{W}\Omega^{\cdot}_{\underline{X}})$$

To see that c^i_Y is a quasi-isomorphism, we can work locally, with the aid of a lifting $(\tilde{X}, \Phi_{\tilde{X}})$ of (X, F_X). Let $(\tilde{Z}, \Phi_{\tilde{Z}}) := (Y \times \tilde{X}, \Phi_Y \times \Phi_{\tilde{X}})$, and let $\tilde{E}$ denote the divided power envelope of X in $\tilde{Z}$. Then we have morphisms:

$$\begin{tikzcd}
Fil^i_X(\mathcal{O}_D \otimes \Omega^{\cdot}_{\tilde{Y}}) \arrow[r] \arrow[dr, "c^i_Y"'] & Fil^i_X(\mathcal{O}_{\tilde{E}} \otimes \Omega^{\cdot}_{\tilde{Z}}) \arrow[d, "c^i_{\tilde{Z}}"] & Fil^i_X(\mathcal{O}_{\tilde{X}} \otimes \Omega^{\cdot}_{\tilde{X}}) \arrow[l] \arrow[dl, "c^i_{\tilde{X}}"] \\
& N^i\mathcal{W}\Omega^{\cdot}_{\underline{X}}. &
\end{tikzcd}$$

The horizontal arrows are quasi-isomorphisms by Theorem 6.7, and $c^i_{\tilde{X}}$ is a quasi-isomorphism by Propositions 3.3 and 7.8. It follows that $c^i_{\tilde{Z}}$ and c^i_Y are also quasi-isomorphisms, as claimed. $\square$

A Technicalities of toric differentials

Let Q be a fine saturated monoid, let R be a regular ring, and let $X := \operatorname{Spec} R[Q]$. Then X is normal, so the complement of its regular locus X_{reg} has codimension at least two. Since the geometric fibers of $X_{\mathrm{reg}} \to \operatorname{Spec} R$ are also regular and X/R is flat, in fact fact X_{reg}/R is smooth, and the sheaves in the complex $\Omega^{\cdot}_{X_{\mathrm{reg}}/R}$ are locally free. The pushforward $j_*\Omega^{\cdot}_{X_{\mathrm{reg}}}$ to all of X is called the complex of *Zariski* or *Danilov* differentials and has been extensively studied. In particular, if $R = \mathbf{C}$, then Danilov [7] showed that the hypercohomology of this complex calculates the singular cohomology of the analytic space associated to X.

If R is flat over $\mathbf{Z}$, these Danilov differentials are the quasi-coherent sheaves associated to the modules $\underline{\Omega}^{\cdot}_{R[Q]/R}$ defined in Definition 2.3, as explained in [20, V.2.1.1.2] and [20, V.2.3.15]. The flatness hypothesis was unfortunately neglected in these assertions, and an example due to Simon Felten [10, Example 7.5] shows that it is not superfluous. (We give a slightly simpler example below.) Although the complex of Danilov differentials also satisfies a Cartier isomorphism [5], we are forced to use instead the complex $\underline{\Omega}^{\cdot}_{R[Q]/R}$, since the notion of "Cartier type," requires commutation with base change, which is not always the case for the Danilov differentials. For more details about this issue, we refer to the errata pages associated to [20], currently available at `https://math.berkeley.edu/~ogus/loggeometryerrata.pdf`.

Example A.1. If $X = \operatorname{Spec} R[Q]$, (with R regular and Q fine and saturated), then its module of Danilov differentials $\tilde{\Omega}_{R[Q]/R}$ is the Q^{gp}-graded submodule of $R[Q] \otimes Q^{\mathrm{gp}}$ which in degree q is the intersection of the submodules $R \otimes F^{\mathrm{gp}}$ as F ranges over the facets of Q containing q, as explained in the course of the proof of [20, V.2.3.13] and in [7, 4.3]. For example, let p be a prime and let Q be the monoid given by generators a, b, c satisfying the relation $a + b = pc$. This monoid is fine and saturated, and its facets F_1 and F_2 are the submonoids generated by a and b respectively. Thus $F_1^{\mathrm{gp}} \cap F_2^{\mathrm{gp}} = \{0\}$, but a and $-b$ become equal in $Q^{\mathrm{gp}} \otimes \mathbf{F}_p$, so $(F_1^{\mathrm{gp}} \otimes \mathbf{F}_p) \cap (F_2^{\mathrm{gp}} \otimes \mathbf{F}_p) = F_1^{\mathrm{gp}} \otimes \mathbf{F}_p$. On the other hand, recall from Definition 2.3 that $\underline{\Omega}^1_{Q/\mathbf{Z}}$ is the Q-graded submodule of $\mathbf{Z}[Q] \otimes Q^{\mathrm{gp}}$

which in degree q is $\langle q\rangle^{\mathrm{gp}}$. Thus we find that, in degree 0, $\underline{\Omega}^1_{\mathbf{F}_p[Q]/\mathbf{F}_p} = 0$ while $\tilde{\Omega}^1_{\mathbf{F}_p[Q]/\mathbf{F}_p} \cong \mathbf{F}_p$.

Although the complex $\underline{\Omega}^{\cdot}_{R[Q]/R}$ cannot be computed as the pushforward of the de Rham complex on the regular locus of $\operatorname{Spec} R[Q]$, it can be viewed as the pushforward of the de Rham complex on a toric resolution of singularities. This will follow from the following result, inspired by ideas of Danilov [6, 1.5].

Proposition A.2. *Let R be a ring, let K be an ideal of a toric monoid Q, and let $f\colon X_K \to X$ be the (normalized) blowup of $X := \operatorname{Spec} R[Q]$ along the ideal $R[K]$. Then the natural map*

$$\underline{\Omega}^{\cdot}_{X/R} \to f_*\underline{\Omega}^{\cdot}_{X_K/R}$$

is an isomorphism. In particular, K can be chosen so that X_K is smooth, in which case $\underline{\Omega}^{\cdot}_{X_K/R} \cong \Omega^{\cdot}_{\underline{X}_K/R}$.

Proof. Choose generators $(k_1, \dots, k_m)$ for K, and for each i, let Q_i be the saturation of the submonoid of Q^{gp} generated by Q and $\{k_j - k_i : j = 1, \dots, m\}$. Then $\{X_i := \operatorname{Spec} R[Q_i] : i = 1, \dots, m\}$ is an open affine cover of X_K, and for each j

$$\Gamma(X_K, \underline{\Omega}^j_{X_K/R}) = \cap_i \{\underline{\Omega}^j_{R[Q_i]/R} : i = 1, \dots, m\}.$$

All these modules are Q^{gp}-graded, and the intersection formula above holds for each degree. In particular, for each $y \in Q^{\mathrm{gp}}$, the degree y part of $\Gamma(X_K, \underline{\Omega}^j_{X_K/R})$ vanishes unless $\underline{\Omega}^j_{R[Q_i]/R,y}$ is nonzero for every i, and, in particular, unless $y \in \cap\{Q_i : i = 1, \dots, m\}$. Assume henceforth that this is the case.

It follows from [20, II,1.7.7] that the map

$$\cup\{\operatorname{Spec} Q_i : i = 1, \dots, m\} \to \operatorname{Spec} Q$$

is surjective. Thus, for every face F of Q, there exist some i and some face F_i of Q_i such that $F_i \cap Q = F$. Then the homomorphism $Q_F \to Q_{i F_i}$ is local, and if F is a facet, also exact [20, 4.2.1]. Since $y \in Q_i \subseteq Q_{i F_i}$, it follows that $y \in Q_F$, and since this is true for every facet of Q and Q is saturated, it follows that $y \in Q$ [20, I,2.4.5].

Now write q for y and let $F := \langle q\rangle$. The strong surjectivity of log blowups [20, II,1.7.7] implies that i and F_i can be chosen so that the map $(Q/F)^{\mathrm{gp}} \to (Q_i/F_i)^{\mathrm{gp}}$ is an isomorphism. This implies that $F^{\mathrm{gp}} = F_i^{\mathrm{gp}}$, and so if q_i is the image of q in Q_i and $\langle q_i\rangle$ is the face of Q_i it generates, that $R \otimes \Lambda^j\langle q\rangle^{\mathrm{gp}} = R \otimes \Lambda^j\langle q_i\rangle^{\mathrm{gp}}$. We conclude that $\underline{\Omega}^j_{R[Q/R],q} = \underline{\Omega}^j_{R[Q_i/R],q}$ and hence that

$$\underline{\Omega}^j_{R[Q/R],q} = \Gamma(X_K, \underline{\Omega}^j_{X_K/R})_q.$$

The fact that K can be chosen to make X_K smooth follows from [18, 5.8], and then $\underline{\Omega}^{\cdot}_{X_K/R} \cong \Omega^{\cdot}_{\underline{X}_K/R}$ by [20, V,2.3.11]. $\square$

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