Midterm I, October 5

Work each problem on a separate sheet of paper. Be sure to put your name, your section number, and your GSI's name on each sheet of paper. Also, at the top of the page, in the center, write the problem number, and be sure to put the pages in order. Write clearly: explanations (with complete sentences when appropriate) will help us understand what you are doing. Note that there are problems on the back of this sheet, for a total of four problems.

1. Consider the following system of linear equations:

$$\begin{aligned} x_1 - x_2 + x_3 &= 5 \\ x_3 - 2x_1 + x_4 &= 2. \end{aligned}$$

   (a) (5 pts) Write the system in matrix form. (You may either write an augmented matrix or a matrix equation.)
   Answer:
   $\left(\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 5 \\ -2 & 0 & 1 & 1 & 2 \end{array}\right)$ or $\begin{pmatrix} 1 & -1 & 1 & 0 \\ -2 & 0 & 1 & 1 \end{pmatrix} X = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

   (b) (5 pts) How many solutions does the system have? Explain your answer. You do not need to find the solutions explicitly.
   Answer: There are infinitely many solutions. It is clear that the two rows of the coefficient matrix are linearly independent, so the equations are consistent. Since there are 4 variables and 2 equations, this means infinitely many solutions.

2. Compute the following matrix operations if they are defined. If not, explain why not.

   (a) (3 pts) $\begin{pmatrix} 1 & 2 \\ 3 & -2 \end{pmatrix} + 3\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$
   Answer: $\begin{pmatrix} 7 & 5 \\ 15 & 4 \end{pmatrix}$

   (b) (3 pts) $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}^{-1}$
   Answer: This is not defined, because for example $2 \cdot 2 - 4 \cdot 1 = 0$.

   (c) (3 pts) $\begin{pmatrix} 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
   Answer: This is not defined because the dimensions do not match.

   (d) (3 pts) $\begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$
   Answer: (8)

   (e) (3 pts) $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}$
   Answer: $\begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$

3. Let

$$A := \begin{pmatrix} 1 & -1 & 2 & 3 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -2 \end{pmatrix}$$

(a) (5 pts) Find the reduced row echelon form of A.
Answer:

$$\begin{pmatrix} 1 & -1 & 2 & 3 \\ 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 2 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) (5 pts) What is the rank of A? What is the dimension of the kernel of the linear transformation T_A corresponding to A? Explain.
Answer: The rank of A is 2, the number of nonzero rows in its reduced row echelon form. The dimension of the kernel is also 2 since it is the number of columns minus the rank.

(c) (5 pts) Find a basis of the image of the linear transformation T_A.
Answer: $\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$

4. Let P_3 denote the vector space of polynomials p of degree at most three. You may use the fact that $\beta := (x^0, x^1, x^2, x^3)$ is a basis of P_3. Note that x^0 is the constant function whose value is always 1. Let T be the linear transformation from P_3 to $\mathbf{R}^2$ taking p to $\begin{pmatrix} p(-1) \\ p(1) \end{pmatrix}$.

(a) (5 pts) What is the dimension of P_3? Explain, recalling the definition of dimension.
Answer: The dimension is 4, since β is a basis for P_3 with 4 elements.

(b) (5 pts) Find a basis for the kernel of T. You do not need to prove that your answer is correct.
Answer: $(x^2 - 1, x^3 - x)$ is one answer.

(c) (5pts) Find the matrix $[T]_\gamma^\beta$ for T with respect to the basis β for P_3 and the standard basis $\gamma := (e_1, e_2)$ for $\mathbf{R}^2$.
Answer: $[T]_\gamma^\beta = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$