Mathematics 54W Professor A. Ogus

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# Math 54 Final Solutions

1. (15 pts) Consider the system of equations

$$\begin{aligned} x_1 - x_2 - x_3 &= -1 \\ 2x_1 + x_2 + 4x_3 &= a \\ x_1 + 2x_2 + 5x_3 &= 2 \end{aligned}$$

(a) For which value(s) of $a$ does there exist a solution?

Solution: We use Gauss elimination:

$$\begin{pmatrix} 1 & -1 & -1 & | & -1 \\ 2 & 1 & 4 & | & a \\ 1 & 2 & 5 & | & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 & | & -1 \\ 0 & 3 & 6 & | & 3 \\ 0 & 3 & 6 & | & a+2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & -1 & | & -1 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & 0 & | & a-1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 2 & | & 1 \\ 0 & 0 & 0 & | & a-1 \end{pmatrix}$$

This has a solution if and only if $a = 1$.

(b) Find all solutions in the above case(s).

Solution: If $a = 1$, then the general solution is $x_1 = -t, x_2 = 1 - 2t, x_3 = t$, for any number $t$.

(c) For $a = -2$, find a solution $(x_1, x_2, x_3)$ which give the closest answer in the sense of least squares.

Solution: A basis for the column space of the matrix above is $(v_1, v_2)$, where

$$v_1 := \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, v_2 := \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix},$$

and Gram-Schmidt gives an orthonormal basis $(w_1, w_2)$, where $w_1 = v_1$ and

$$w_2 = v_2 - \frac{3}{6} v_1 = \begin{pmatrix} -3/2 \\ 0 \\ 3/2 \end{pmatrix}$$

The orthogonal projection of $v := \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix}$ on the column space is then

$$\frac{-1}{2} w_1 + 1 w_2 = \frac{-1}{2} v_1 + v_2 - \frac{1}{2} v_1 = -v_1 + v_2$$

Hence one answer is $x_1 = -1, x_2 = 1$. One can add to this any element of the kernel.

2. (15 pts). Write the definition of each of the following concepts. Use complete sentences and be as precise as you can.

   (a) A *linear subspace* of a vector space.
   A linear subspace of a vector space $V$ is a nonempty subset which is closed under addition and scalar multiplication.

   (b) A *linearly independent sequence* in a vector space $V$.
   A sequence $(v_1, \dots, v_n)$ is linearly independent if the equation $c_1 v_1 + \dots + c_n v_n = 0$ implies that each $c_i = 0$.

   (c) A *basis* for a vector space $V$.
   A basis for $V$ is a sequence which is linearly independent and which spans $V$.

   (d) The *dimension* of a vector space. State the theorem(s) which makes this definition meaningful.
   Every vector space has a basis, and any two such have the same cardinality. This cardinality is called the dimension of the space.

   (e) The *orthogonal projection* of a vector in an inner product space $V$ onto a (finite dimensional) linear subspace $W$ of $V$.
   This orthogoal projection of $v$ on $W$ is the unique element $w$ of $W$ such that $v - w$ is orthogonal to every element of $W$.

   (f) A *linear transformation* $T$ from a vector space $V$ to a vector space $W$.
   A linear transformation from $V$ to $W$ is a function $T$ from $V$ to $W$ such that $T(av_1 + bv_2) = aT(v_1) + bT(v_2)$ for all $a, b, v_1, v_2$.

   (g) An *eigenvector* and an *eigenvalue* of a linear operator $T: V \to V$.
   An eigenvector of $T$ is an element $v$ such that $T(v) = \lambda v$ for some scalar $\lambda$. An eigenvalue of $T$ is a scalar $\lambda$ for which there exists some nonzero $v$ with $T(v) = \lambda v$.

3. (15 pts) Let $W$ denote the set of functions of the form $e^{2x} f(x)$, where $f(x)$ is a polynomial of degree less than or equal to 2. If $f \in W$, let $T(f) := f'$.

   (a) Find the matrix $A$ representing $T$ with respect to the basis $\{e^{2x}, xe^{2x}, x^2e^{2x}\}$.
   Solution: $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

   (b) Find the eigenvectors and eigenvalues of $A$. Is $A$ diagonalizable?
   Solution: This is clear by "inspection." The only eigenvalue is 2, and the eigenspace is spanned by $e^{2x}$. The operator is not diagonalizable.

4. (15 pts) Let $A$ and $B$ be as follows:

$$A = \begin{pmatrix} 2 & -2 & 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 2 & 1 & 4 \\ 3 & -3 & 2 & 5 & 1 & 9 \end{pmatrix} \quad B = \begin{pmatrix} 1 & -1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

I think these matrices are row equivalent. Assuming this is the case, answer the following questions.

(a) Find a basis for the nullspace of $A$
Solution: Since $NS(A) = NS(B)$, we get a basis element for each free column:

$$\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) Find a basis for the column space of $A$ from among the columns of $A$, omitting redundant columns from left to right.
Solution: We choose the columns of $A$ corresponding to pivot columns of $B$:

$$\begin{pmatrix} 2 \\ 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(c) Find an $n \times 4$ matrix $C$ of rank $n$ whose nullspace is the column space of $B$.
Since the column space of $B$ consists of those vectors whose last coordinate is zero, $C = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$ will do.

5. (10 pts.) Let $A = \begin{pmatrix} 4 & -3 \\ 2 & -3 \end{pmatrix}$.

(a) Find a diagonal matrix $D$ which is similar to $A$.
Solution: $f_A(t) = t^2 - t + -6 = (t-3)(t-2)$. Thus $A$ has distinct eigenvalues 3 and $-2$, so $D := \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$ will work.

(b) Find an invertible matrix $S$ such that $A = SDS^{-1}$.
Solution: $S = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ will do.

6. (10 pts) Consider the system of differential equations:

$$\begin{aligned} f' &= 4f - 3g & (1) \\ g' &= 2f - 3g & (2) \end{aligned}$$

(a) Write this system in matrix form.
Solution: This is just $Y' = AY$, where $A := \begin{pmatrix} 4 & 3 \\ 2 & -3 \end{pmatrix}$.

(b) Find a fundamental solution set for the equation in part (a). Hint: you may find problem 5 useful.
Solution: We can use the eigenvectors: if $Av = \lambda v$, then $e^{\lambda t}v$ is a solution. In particular,

$$Y_1(t) := e^{-2t}\begin{pmatrix}1\\2\end{pmatrix} \quad , Y_2(t) = e^{3t}\begin{pmatrix}3\\1\end{pmatrix}$$

will do.

(c) Compute the Wronskian of your solution set.
Solution: We know by Abel's theorem that the Wronskian is $W = ce^t$ for some constant $c$, and evaluating at 0 we see that $c = -5$.

(d) Find a pair of functions $f, g$ satisfying the equations (1) and (2) and such that $f(0) = 2$ and $g(0) = -1$.
Solution: We want $Y = a_1Y_1 + a_2Y_2$ to satisfy $Y(0) = \begin{pmatrix}2\\-1\end{pmatrix}$.
Thus we need

$$a_1\begin{pmatrix}1\\2\end{pmatrix} + a_2\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}.$$

Clearly $a_1 = -1$ and $a_2 = 1$ will do. Thus we get

$$\begin{pmatrix}f\\g\end{pmatrix} = -e^{-2t}\begin{pmatrix}1\\2\end{pmatrix} + e^{3t}\begin{pmatrix}3\\1\end{pmatrix},$$

so

$$f = -e^{2t} + 3e^{3t}, \quad g = -2e^{-2t} + e^{3t}.$$

7. (15 pts) Existence and uniqueness theorem

(a) Carefully state the existence and uniqueness theorem for a linear second order differential equation.
Solution: Let $I \subseteq \mathbf{R}$ be an open interval, and let $p, q$, and $g$ be continuous functions on $I$. Then given any two real numbers $a$ and $b$ and any $t_0 \in I$, there is a unique function $y$ on $I$ such that $y'' + py' + qy = g$ and $y(t_0) = a$ and $y'(t_0) = b$.

(b) Consider the space $W$ of all twice continuously differentiable functions functions $y : \mathbf{R} \to \mathbf{R}$ such that $x^2y'' - 2xy' + 2y = 0$. Find a basis for $W$. (Hint: try functions of the form $y = x^\lambda$.) Use the existence and uniqueness theorem to prove that your set really is a basis, at least for positive values of $x$.
Solution: If we try $y = x^\lambda$, we find that the equation becomes

$$x^2\lambda(\lambda - 1)x^{\lambda-2} - 2x\lambda x^{\lambda-1} + 2x^\lambda = 0.$$

This holds if $\lambda^2 - 3\lambda + 2 = 0$, that is, if $\lambda = 2$ or $\lambda = 1$. Thus $x$ and $x^2$ satisfy the equation. They are clearly linearly independent. The theorem doesn't apply on all of $\mathbf{R}$ because of the singularity at 0, but it does apply on the intervals $I^+ := \{x : x > 0\}$ and $I^- := \{x : x < 0\}$, where the functions remain linearly indpendent. Now if $f$ satisfies the function on all of $\mathbf{R}$, then the uniqueness theorem implies that there exist real numbers $a, b$ such that $f(x) = ax + bx^2$ for $x > 0$ and also $c, d$ such that $f(x) = cx + dx^2$ for $x < 0$. Since $f$ is continuously differentiable at 0, $a = c$, and since its second derivative is also continuous, $d = b$. Thus $x$ and $x^2$ span the solution space.

(c) Compute the Wronskian of your basis. What is its value at zero? Why doesn't this contradict the existence and uniqueness theorem?
Solution: The Wronskian is $\det \begin{pmatrix} x & x^2 \\ 1 & 2x \end{pmatrix} = x^2$. It vanishes at zero, but this is okay because the theorem doesn't apply at 0.

8. (15 pts) (Separation of variables and the heat equation.)

(a) Write the partial differential equation that describes the behaviour of the temperature distribution $u(x, t)$ of a uniform bar of length $\ell$.
Solution: This is

$$\frac{\partial u}{\partial t}(x, t) = \beta \frac{\partial^2 u}{\partial x^2}(x, t),$$

where $\beta$ is a positive constant.

(b) Write the boundary values $u(x, t)$ satisfies if the left end is insulated and the right end is held at temperature $0^o$.
Solution: This is

$$\frac{\partial u}{\partial x}(0, t) = 0, u(\ell, t) = 0.$$

(c) Use separation of variables to find an infinite dimensional family of solutions to the differential equation satisfying the boundary value problem you just described. Explain what you are doing, but you do not need to give complete proofs, and shortcuts are allowed where the results are similar to those we did in class. You do not need to check that these are in any sense "all" the solutions.
Solution: We write $u(x, t) = X(x)T(t)$, so the equations become

$$T'(t)X(x) = \beta X''(x)T(t), X'(0)T(t) = 0, X(\ell)T(t) = 0.$$

Hence

$$\frac{T''(t)}{\beta T(t)} = \frac{X''(x)}{X(x)} = K$$

for some constant $K$. We expect to get solutions to the boundary values only if $K$ is negative, say $K = -r^2$, so that a general solution is of the form $a \cos(rx) + b \sin(rx)$. The first boundary condition then says $-ar \sin(r0) + br \cos(r0) = 1$, which implies that $br = 0$, so if $r \neq 0$, $b = 0$, and the second says that $a \cos(r\ell) = 0$. This holds if $r\ell = n\pi/2$ for some odd integer $n$. Thus we get solutions to the original problem:

$$u_n(x, t) = e^{\frac{-\beta n^2 \pi^2 t}{\ell^2}} \cos(\frac{n\pi x}{\ell})$$

(for $n$ an odd integer).

9. (15 pts) Suppose $f\colon \mathbf{R} \to \mathbf{R}$ is odd, periodic with period $2\pi$, and satisfies $f(x) = \cos x$ for $x \in (0, \pi)$.

   (a) Draw a sketch of the graph of $f$, labeling your axes carefully.

   (b) Find a Fourier series which represents $f$. Explain how you find the coefficients. You may use one or more of the formulas at the end of the test to evaluate the coefficients if you like.
   Solution: We need to consider the sine expansion, since $f$ is odd. Thus, we want

$$F(x) = \sum_n a_n \sin(nx), \quad \text{where } a_n = \frac{2}{\pi}\int_0^\pi \cos(x)\sin(nx).$$

   Using the formulas at the end, we find that

$$\begin{aligned}
\frac{2}{\pi}\int_0^\pi \cos(x)\sin(nx) &= = \frac{2}{\pi}\int_0^\pi \frac{\sin((1+n)x) + \sin((n-1)x)}{2} \\
&= \frac{-1}{\pi}\left(\frac{\cos((1+n)x)}{1+n} + \frac{\cos(n-1)x)}{1-n}\right)\Big|_0^\pi \\
&= \frac{1}{\pi}\left(\frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1}\right) + \frac{1}{\pi}\left(\frac{1}{n+1} + \frac{1}{n-1}\right)
\end{aligned}$$

   This is $\frac{4n}{\pi(n^2-1)}$ if $n$ is even and vanishes if $n$ is odd. (The case $n = 1$ has to be checked separately, but gives the same answer.) So the answer is

$$F(x) = \sum_{n \text{ even}} \frac{4n}{\pi(n^2-1)} \sin(nx).$$

   (c) Find the limit of the Fourier series at each $x \in \mathbf{R}$.
   Solution: Since $f$ and its derivative are piecewise continuous, the limit $F(x)$ is given by

$$F(x) = \frac{f(x^+) + f(x^-)}{2}$$

   This is $f(x)$ if $x \in (0, \pi)$ and is 0 if $x$ is an integral multiple of $\pi$.

10. (15 pts) Suppose that a bar of length $\pi$ with thermal coefficient $\beta = 3$ is insulated everywhere along its surface, except perhaps at the endpoints.

   (a) If the ends of the bar are kept at $0^o$ when $t > 0$ and the initial temperature distribution is $u(x, 0) = \sin(3x)$, find a formula for the temperature distribution at all times.
   Solution: $u(x, t) = e^{-27t}\sin(3x)$.

(b) If instead the left end is kept at $0^o$ and the other at $2\pi^o$, what is the limiting temperature distribution (steady state solution) of the temperature as $t \to \infty$? Verify directly that this limit distribution satisfies the heat equation.

Solution: $v(x,t) = v(x) = 2x$ satisfies the boundary contitions and also

$$\frac{\partial v}{\partial t} = 0 = 3\frac{\partial^2 v}{\partial x^2}$$

(c) In the situation (b), assume again that the initial temperature is given by $u(x,0) = \sin(3x)$. Find a formula for the temperature distribution at time $t$. You may use one or more of the formulas listed at the end of the test.

Solution: We know that $u(x,t) = v(x) - w(x,t) + e^{-27t}\sin(3x)$, where $v$ is the steady state solution and $w$ is the solution corresponding to the initial condition $v(x)$. Using the formula at the end, we see that the Fourier sine coefficients of $v(x)$ are given by

$$a_n = \frac{4(-1)^{k+1}}{n}.$$

Hence

$$u(x,t) = 2x - \sum_{n=1}^{\infty} \frac{4(-1)^{k+1}}{n} \sin(nx) + e^{-27t}\sin(3x)$$

## Formulas

$$
\begin{aligned}
1 &= 2\sum \frac{1+(-1)^{k+1}}{\pi k}\sin(kx) \quad \text{for } 0 < x < \pi \\
x &= 2\sum \frac{(-1)^{k+1}}{k}\sin(kx) \quad \text{for } 0 \le x < \pi
\end{aligned}
$$

$$
\begin{aligned}
\sin(nx)\cos(x) &= \frac{\sin((1+n)x) + \sin((n-1)x)}{2} \\
\cos(nx)\sin(x) &= \frac{\sin((1+n)x) + \sin((1-n)x)}{2}
\end{aligned}
$$