Vanishing of Cohomology

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Proposition 1 *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $(T^i, \delta^i)\colon \mathcal{A} \to \mathcal{B}$ be a cohomological δ-functor. Suppose that $\mathcal{F}$ is a full subcategory of $\mathcal{A}$ with the following properties:*

1. *For every exact sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ in $\mathcal{A}$ with F_1 and F_2 in of $\mathcal{F}$, then F_3 also belongs to $\mathcal{F}$ and the sequence*

$$0 \to T^0(F_1) \to T^0(F_2) \to T^0(F_3) \to 0$$

is exact.

2. *For $i > 0$, the functors T^i are effaceable in $\mathcal{F}$.*

Then $T^i(F) = 0$ for every $F \in \mathcal{F}$ and every $i > 0$.

Proof: If F is an object of $\mathcal{F}$, then since T^1 is effaceable in $\mathcal{F}$, there exists an embedding $\epsilon\colon F \to \tilde{F}$ where $\tilde{F} \in \mathcal{F}$ and $T^1(\epsilon) = 0$. Let $a\colon \tilde{F} \to Q$ be the cokernel of ϵ. We have an exact sequence

$$T^0(F) \longrightarrow T^0(\tilde{F}) \xrightarrow{a} T^0(Q) \xrightarrow{\delta} T^1(F) \xrightarrow{0} T^1(\tilde{F})$$

Since F and $\tilde{F}$ belong to $\mathcal{F}$, hypothesis (1) implies that a is surjective, so $\delta = 0$, and it follows that $T^1(F) = 0$. We proceed to prove that $T^i(F) = 0$ for all F and all $i > 0$ by induction on i. Assume this is true for i and that F is any object of $\mathcal{F}$. Since T^{i+1} is effaceable in $\mathcal{F}$, there exists an injection $\epsilon\colon F \to \tilde{F}$ with $\tilde{F} \in \mathcal{F}$, and with $T^{i+1}(\epsilon) = 0$. By hypothesis (1) , the cokernel Q of ϵ belongs to $\mathcal{F}$, and by the induction hypothesis, $T^i(Q) = 0$. Then the exact sequence

$$T^i(Q) \longrightarrow T^{i+1}(F) \xrightarrow{0} T^{i+1}(\tilde{F})$$

shows that $T^{i+1}(F) = 0$. □

Corollary 2 *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $(T^{\cdot}, \delta^{\cdot})$ a cohomological δ-functor from $\mathcal{A}$ to $\mathcal{B}$. Suppose that $\mathcal{A}$ has enough injectives and that T^i is effaceable for all $i > 0$. Then $T^i(I) = 0$ for all $i > 0$ and every injective object I of $\mathcal{A}$.*

Proof: We apply the previous argument with $\mathcal{F}$ the category of injective objects of $\mathcal{A}$. If $0 \to F_1 \to F_2 \to F_3 \to 0$ is an exact sequence in $\mathcal{A}$ with F_1 and F_2 injective, then the sequence splits. It follows that F_3 is injective and that $T^0(F_2) \to T^0(F_3)$ is surjective. Furthermore, if $A \in \mathcal{A}$ and $i > 0$, then there exists a monomorphism $\epsilon \colon A \to \tilde{A}$ with $T^i(\epsilon) = 0$, since T^i is effaceable. Since $\mathcal{A}$ has enough injectives, there exists another monomorphism $\tilde{A} \to F$ with F injective. Then the composite $\epsilon' \colon A \to \tilde{A} \to F$ is a monomorphism and $T^i(\epsilon') = 0$. □

Corollary 3 *If X is a topological space and F is a flasque abelian sheaf on X, then $H^q(X, F) = 0$ for $q > 0$.*

Proof: We will use the following result.

Lemma 4 *If G is a flasque abelian sheaf on a topological space X, then any G-torsor on X has a global section.*

Proof: Let T be a G-torsor, *i.e.*, a sheaf of F-sets on X such that the stalks are nonempty and such that the map $F \times T \to T \times T$ is bijective. Consider the set of pairs (U, t) such that U is open in X and $t \in T(U)$, and write $(U_1, t_1) \geq (U_2, t_2)$ if $U_2 \subseteq U_1$ and t_2 is restriction of T_1 to U_2. If $\mathcal{U} := \{(U_\lambda, t_\lambda) : \lambda \in \Lambda\}$ is any chain in this ordered set, then the fact that T is a sheaf guarantees that there is a unique $t \in T(\cup_\lambda U_\lambda)$ whose restriction to each U_λ is t_λ, and then $(\cup U_\lambda, t)$ is an upper bound for $\mathcal{U}$. The Hausdorff maximality principle then guarantees the existence of a maximal pair (U, t), and it suffices to prove that $U = X$. Otherwise there exists $x \in X \setminus U$, and since T is a torsor, there exist an open neighborhood V of x and an $s \in T(V)$. Then there exists a unique $g \in G(U \cap V)$ such that $gs_{|U \cap V} = t_{|U \cap V}$. Since G is flasque, there is an $h \in G(V)$ such that $h_{|U \cap V} = g$. Since T is a sheaf, there is a (unique) section of T on $U \cap V$ whose restriction to U is t and whose restriction to V is hs. □

We can now verify that the category $\mathcal{F}$ of flasque abelian sheaves on X satisfies the hypotheses of Proposition 1 If $0 \to F' \to F \to F'' \to 0$ is an exact sequence in with F' and F flasque, then the lemma implies that then for every open set U of X, the map $F(U) \to F''(U)$ is surjective. Thus the rows of the commutative diagram below are exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F'(X) & \longrightarrow & F(X) & \longrightarrow & F''(X) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho''} & & \\
0 & \longrightarrow & F'(U) & \longrightarrow & F(U) & \longrightarrow & F''(U) & \longrightarrow & 0
\end{array}$$

In this diagram ρ is surjective because F is flasque and it follows that ρ'' is surjective. Thus F'' is flasque, and the category of flasque sheaves satisfies (1.1). Since every injective is flasque, it also satisfies (1.2). □

Theorem 5 *Suppose that X is a topological space and $\mathcal{B}$ is a base for its topology which is closed under finite intersection and such that each $U \in \mathcal{B}$ is quasi-compact. Let F be a sheaf of abelian groups on X. Then the following are equivalent:*

1. *For every $U \in \mathcal{B}$, $H^q(U, F) = 0$ for $q > 0$.*

2. *For every finite open cover $\mathcal{U} \subseteq \mathcal{B}$ of an element U of $\mathcal{B}$, the Cech cohomology $\check{H}^q(\mathcal{U}, F)$ of F with respect to $\mathcal{U}$ vanishes.*

Proof: We omit the proof that (1) implies (2). To prove that (2) implies (1), consider the set $\mathcal{F}$ of all abelian sheaves on X satisfying (2). We claim that if $F \in \mathcal{F}$ then $H^q(U, F) = 0$ for $q > 0$ and $U \in \mathcal{B}$. Without loss of generality, we may assume that $X \in \mathcal{B}$, and it will suffice to prove that $H^q(X, F) = 0$ for $q > 0$. By Hartshorne (II 4.3), $\mathcal{F}$ contains all flasque sheaves, and in particular all injective sheaves. So by Proposition 1, it will suffice to prove that $\mathcal{F}$ satisfies (1.1). Suppose that $0 \to A \to B \to C \to 0$ is an exact sequence of abelian sheaves on X and A and B belong to $\mathcal{F}$. If $\mathcal{U} \subseteq \mathcal{B}$ is a finite open cover of an element U of $\mathcal{B}$, then by hypothesis the Cech cohomology groups $\check{H}^q(\mathcal{U}, A)$ vanish for $q > 0$, and in particular for $q = 1$. Since any $U \in \mathcal{B}$ is quasi-compact, it follows that any A-torsor on U is trivial, and hence that the sequence

$$0 \to A(U) \to B(U) \to C(U) \to 0$$

is exact. Now if $\mathcal{U}$ is any finite subset of $\mathcal{B}$, it follows that for any multi-index I and any $I \to \mathcal{U}$, the intersection U_I belongs to $\mathcal{B}$, and hence the sequence

$$0 \to \prod_I A(U_I) \to \prod_I B(U_I) \to \prod_I C(U_I) \to 0$$

is exact. In other words, we get an exact sequence of complexes:

$$0 \to \check{C}^{\cdot}(\mathcal{U}, A) \to \check{C}^{\cdot}(\mathcal{U}, B) \to \check{C}^{\cdot}(\mathcal{U}, C) \to 0.$$

Taking the long exact sequence of cohomology we find the exact sequence

$$\check{H}^q(\mathcal{U}, B) \to \check{H}^q(\mathcal{U}, C) \to \check{H}^{q+1}(\mathcal{U}, A).$$

Since A and B belong to $\mathcal{F}$, we deduce that $\check{H}^q(\mathcal{U}, C) = 0$ for $q > 0$ if $\mathcal{U}$ is a cover of an element of $\mathcal{B}$. □

Theorem 6 *If X is an affine scheme and F is a quasi-coherent sheaf on X, then $H^q(X, F) = 0$ for $q > 0$.*

Proof: Thanks to the previous result, it will suffice to show that if $\mathcal{B}$ is the set of special affine open subsets of X and $\mathcal{U}$ is any finite cover of an element U of $\mathcal{B}$, then the Cech cohomology $\check{H}^q(U, F) = 0$ for $q > 0$. Note first that if $j: U \to X$ is the inclusion map, then $j_* j^* F$ is quasi-coherent on X, because the j is a quasi-compact and quasi-separated map. The same applies to the inclusion of any U_I, and since $\mathcal{U}$ is finite, we see that all the terms of the "sheaf" Cech complex $\underline{C}^{\cdot}(\mathcal{U}, F)$ are quasi-coherent. This complex thus defines a resolution of F by quasi-coherent sheaves, and since the global section functor is exact on the category of quasi-coherent sheaves, the complex remains exact when we apply Γ. Thus the global Cech complex is acyclic, and the result is proved. □