Differentials and Smoothness

February 17, 2017

We present a supplement, and in some cases alternative, to Hartshorne's Chapter II,§8 and Chapter II,§10. These notes are not meant to be self-contained, and will required reading Hartshorne for some definitions, some statements, and many important examples. We often do not include detailed proofs.

1 Derivations and deformations

Definition 1 *Let n be a natural number. An "nth order thickening" is a closed immersion $i\colon S \to T$ which is defined by an ideal $\mathcal{I}$ such that $\mathcal{I}^{n+1} = 0$. A "nilpotent thickening" is a closed immersion which is an nth order thickening for some $n > 0$.*

If Y is a scheme, a thickening in the category of Y-schemes means a thickening $i\colon S \to T$ of Y-schemes. Sometimes to emphasize this we may want to write $i\colon S/Y \to T/Y$.

The underlying map of topological spaces of a nilpotent thickening is a homeomorphism, and we will sometimes identify the underlying spaces of S and T. First order thickening are especially convenient to work with. If $\mathcal{I}$ is the ideal of a first order thickening $i\colon S \to T$, then $\mathcal{I}^2 = 0$, and this means that if a is a local section of $\mathcal{O}_T$ and x of $\mathcal{I}$, then the product ax depends only on the image of a in $\mathcal{O}_S$. Thus $\mathcal{I}$ has a natural structure of an $\mathcal{O}_S$-module: the natural map $\mathcal{I} \to i_*i^*(\mathcal{I})$ is an isomorphism. We will sometimes ourselves to identify $\mathcal{I}$ with $i^*(\mathcal{I})$.

Example 2 *If $\mathcal{M}$ is a quasi-coherent sheaf of $\mathcal{O}_S$-modules, there is a scheme $D(\mathcal{M})$ whose underlying topological space is S and whose structure sheaf* *is $\mathcal{O}_S \oplus \mathcal{M}$, with the obvious $\mathcal{O}_S$-module structure and with multiplication defined by $(a,x)(b,y) := (ab, ay+bx)$. Then the map $(a,x) \to a$ defines a first order thickening $i: S \to D(\mathcal{M})$ whose ideal sheaf $\mathcal{I}$ identifies with $\mathcal{M}$. The map $\mathcal{O}_S \to \mathcal{O}_S \oplus \mathcal{M}$ sending a to $(a,0)$, defines a morphism $\rho: D(\mathcal{M}) \to S$ such that $\rho \circ i = \mathrm{id}$.*

Definition 3 *Let F be a presheaf on the category of Y-schemes, let $i: S \to T$ be a nilpotent thickening, and let ξ be an element of $F(S)$. Then a "deformation of ξ to T" is an element ζ of $F(T)$ such that $F(i)(\zeta) = \xi$, and we write $\mathrm{Def}_\xi(T)$ to denote the set of all such elements.*

If $T' \to T$ is any morphism and $S' := S \times_T T'$, then $S' \to T'$ is an nth order thickening, and a deformation of ξ to T pulls back to a deformation of the pullback ξ' to S' to T'. Thus Def_ξ becomes a presheaf on the category of T-schemes. If F is a sheaf, then Def_ξ defines a sheaf on the Zariski topology of T, equivalently on the Zariski topology of S.

If a thickening $S \to T$ admits a retraction $\rho: T \to S$ (as in Example 2), then $F(\rho)(\xi)$ is automatically a deformation of ξ.

Definition 4 *Let $f: X \to Y$ be a morphism of schemes and let $\mathcal{E}$ be a sheaf of $\mathcal{O}_X$-modules. Then $\mathrm{Der}_{X/Y}(\mathcal{E})$ is the sheaf of $f^{-1}(\mathcal{O}_Y)$-linear maps $D: \mathcal{O}_X \to \mathcal{E}$ such that $D(ab) = D(a)b + aD(b)$ for any two local sections a, b of $\mathcal{O}_X$.*

Note that $\mathrm{Der}_{X/Y}$ is an $\mathcal{O}_X$-submodule of $\mathrm{Hom}(\mathcal{O}_X, \mathcal{E})$.

The following notion is extremely pervasive in mathematics.

Definition 5 *Let X be a topological space, let $\mathcal{G}$ be a sheaf of groups on X. A (left) $\mathcal{G}$-*pseudo torsor *is a sheaf of (left) $\mathcal{G}$-sets: $\mathcal{G} \times \mathcal{S} \to \mathcal{S}$ such that the corresponding map*

$$\mathcal{G} \times \mathcal{S} \to \mathcal{S} \times \mathcal{S} : (g,s) \mapsto (gs, s)$$

is an isomorphism. A torsor *is a pseudo-torsor each of whose stalks is nonempty.*

There is an obvious notion of a morphism of $\mathcal{G}$-torsors, and thus an obvious notion of the category of $\mathcal{G}$-torsors on X and of the set of isomorphism classes of $\mathcal{G}$-torsors. If $\mathcal{S}$ is a $\mathcal{G}$-torsor and U is an open subset of X and if $\mathcal{S}(U)$ is not empty, then $\mathcal{S}(U)$ is isomorphic to $\mathcal{G}(U)$: given an $s \in \mathcal{S}(U)$, the map $\mathcal{G}(U) \to \mathcal{S}(U) : g \mapsto gs$ is an isomorphism of left $\mathcal{G}(U)$-sets.

Theorem 6 *Let $f: X \to Y$ be a morphism of schemes and let X/Y be the functor h_X from the category of Y-schemes to the category of sets. Let $i: S \to T$ be a first-order thickening, defined by an ideal $\mathcal{I}$, and let $g: S \to X$ be an element of $X/Y(S)$. Then the sheaf $g_*(\mathrm{Def}_g)$ has a natural structure (see the formula below) of a pseudo-torsor under the sheaf of groups*

$$\mathrm{Der}_{X/Y}(\mathcal{O}_X, g_*\mathcal{I}) \cong \mathcal{H}om(\Omega^1_{X/Y}, g_*\mathcal{I}).$$

Proof: A deformation h of g is a morphism $h: T \to X$ such that $h \circ i = g$. Since i is a homeomorphism and g is given, to give h is the same as to give a homomorphism $h^\sharp: \mathcal{O}_X \to g_*\mathcal{O}_T$. Let $D: \mathcal{O}_X \to g_*\mathcal{I}$ be a derivation, and define $\tilde{h}^\sharp: \mathcal{O}_X \to g_*\mathcal{O}_T$ to be $D + h^\sharp$. This map is $\mathcal{O}_Y$-linear, and we claim that in fact it is a homomorphism. If $a, b \in \mathcal{O}_X$,

$$(D + h^\sharp)(ab) = D(ab) + h^\sharp(ab) = aDb + bDa + h^\sharp(a)h^\sharp(b),$$

while

$$(D + h^\sharp((a)(D + h^\sharp(b)) = D(a)D(b) + h^\sharp(b)D(a)h^\sharp(a)D(b) + h^\sharp(a)h^\sharp(b).$$

Since $\mathcal{I}^2 = 0$, $D(a)D(b) = 0$, and $h^\sharp(b)D(a) = g^\sharp(b)D(a) = bD(a)$; and similarly $h^\sharp(a)D(b) = g^\sharp(a)D(b) = aD(b)$. and D really is a derivation.

On the other hand, if h_1 and h_2 are deformations of g, then the $\mathcal{O}_Y$-linear map $h_2^\sharp - h_1^\sharp$ factors through $\mathcal{I}$, and it is easy to check that this difference is a derivation $\mathcal{O}_X \to g_*\mathcal{I}$. □

For example if $\mathcal{M}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, we have a standard deformation $\rho \in \mathrm{Def}_{\mathrm{id}}(D(\mathcal{M})$, and if $D \in \mathrm{Der}_{X/Y}(\mathcal{M})$, then $D + \rho$ is the deformation given by $a \mapsto (a, Da) \in \mathcal{O}_X \oplus \mathcal{M}$.

Corollary 7 *Let $f: X \to Y$ be a morphism of schemes, and let $\mathcal{I}_{X/Y}$ be the ideal of the (locally closed) diagonal embedding: $\Delta: X \to X \times_Y X$. Then the map*

$$d: \mathcal{O}_X \to \mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y} : a \mapsto p_2^\sharp(a) - p_1^\sharp(a)$$

is a universal derivation.

Proof: Let T be the locally closed subscheme of $X \times_Y X$ defined by $\mathcal{I}^2_{X/Y}$. Then $X \to T$ is a first order thickening, and p_2 and p_1 are two deformations of $\mathrm{id}X$. It follows that their difference d is a derivation. Suppose $D: \mathcal{O}_X \to$

$\mathcal{M}$ is any derivation of X/Y; we claim that D factors through a unique $\mathcal{O}_X$-linear map $\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y} \to \mathcal{M}$. We already know that the target of the universal derivation is quasi-coherent, so we may assume that $\mathcal{M}$ is also quasi-coherent. Then the homomorphism $\mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{M} : a \mapsto (a, Da)$ defines a morphism of schemes $\xi\colon D(\mathcal{M}) \to X$, and the pair (ρ, ξ) defines a morphism $D(\mathcal{M}) \to X \times_Y X$. These morphisms agree on $X \subseteq D(\mathcal{M})$ and hence sends $\mathcal{I}_{X/Y}$ to the ideal $\mathcal{I} = (0, \mathcal{M})$ of X in $D(\mathcal{M})$. Since $\mathcal{I}^2 = 0$, this morphism indeces a map $h\colon \mathcal{I}_{X/Y}\mathcal{I}^2_{X/Y} \to \mathcal{I}$. Moreover, if $a \in \mathcal{O}_X$, $h(p_2^\sharp(a) - p_1^\sharp(a)) = \xi^\sharp(a) - \rho^\sharp(a) = D(a)$.

To check the uniqueness of h, let us observe that $p_{1*}\mathcal{I}_{X/Y}/\mathcal{I}^2_{X/Y}$ is generated as a sheaf of $\mathcal{O}_X$-modules the image of d. Working locally, we may assume that X/Y is given by a homomorphism $R \to A$. Then $\mathcal{I}_{X/Y}$ is the sheaf associated with the ideal $K := \mathrm{Ker}(A \otimes_R A \to A$. Say $\sum a_i \otimes b_i \in K$. Then $\sum a_i b_i = 0$, so

$$\sum a_i \otimes b_i = \sum a_i \otimes b_i - \sum a_i b_i \otimes 1 = \sum a_i (1 \otimes b_i - b_i \otimes 1).$$

$\square$

The exact sequences in the following result help with the calculation of sheaves of differentials. The proof is an easy consequence of the univeral mapping property of differentials.

Theorem 8 *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphims of schemes and let $h := g \circ f$.*

1. *The natural maps fit into an exact sequence:*

$$g^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$$

2. *if f is a closed immersion defined by an ideal sheaf $\mathcal{I}$, $\Omega_{X/Y} = 0$, and there is an exact sequence:*

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0$$

where δ fits into a commutative diagram:

$$\begin{tikzcd}
\mathcal{I} \arrow[r, "d"] \arrow[d] & \Omega_{Y/Z} \arrow[d] \\
\mathcal{I}/\mathcal{I}^2 \arrow[r, "\delta"] & f^*\Omega_{Y/Z}
\end{tikzcd}$$

3. If $b\colon Y' \to Y$ is a morphism and $X' := Y' \times_Y X$, the natural map $pr^*\Omega^1_{X/Y} \to \Omega^1_{X'/Y'}$ is an isomorphism.

Example 9 In the situation of part (2) of Theorem 8, suppose that $X = Z$ and that $h = \mathrm{id}_X$. Then the sequence (2) induces an isomorphism

$$\mathcal{I}/\mathcal{I}^2 \to f^*\Omega_{Y/Z}.$$

Indeed, it is clear that $\Omega_{X/X} = 0$, and it remains only to prove that the map

$$\delta\colon I/I^2 \to i^*\Omega_{X/Y}$$

is injective. We will construct an inverse of this map as follows. Let Y_1 be the closed subscheme of Y defined by $\mathcal{I}^2$, and let $h\colon Y_1 \to Y$ and $f_1\colon X \to Y_1$ be the natural inclusions, and let $g_1 = g \circ h\colon Y \to X$. Note that f_1 is a first-order thickening in the category of X-schemes, since $g_1 \circ f_1 = \mathrm{id}_X$. Let $\tilde{h} := f \circ g \circ h\colon Y_1 \to Y$, and note that $\tilde{h} \circ f = f$ That is, h and $\tilde{h}$ are two deformations of f to Y_1, and hence $h^\sharp - \tilde{h}^\sharp$ is a derivation $\mathcal{O}_Y \to \mathcal{I}/\mathcal{I}^2$. If $a \in \mathcal{I}$, then

$$D(a) = h^\sharp(a) - \tilde{h}^\sharp(a) = h^\sharp(a) - h^\sharp)g^\sharp(f^\sharp(a)) = h^\sharp(a)$$

Thus D defines a homomorphism $\Omega^1_{Y/X}$ sending da to the class of a in $\mathcal{I}/\mathcal{I}^2$ and provides a splitting of the map δ.

For example, suppose that Y is a scheme of finite type over a field k and the inclusion $f\colon X \to Y$ is corresponds to a k-rational point x of Y. Then we get an isomorphism:

$$m_x/m_x^2 \cong \Omega_{Y/k}(x).$$

Thus in this case the Zariski tangent space of Y at x, the k-dual of m_x/m_x^2, Equivalently the set of deformations of the inclusion $x \to Y$ to the dual numbers $D_k(\epsilon)$, becomes identified with the set of maps $\Omega_{Y/k} \to i_*k(x)$, that is, with the fiber of $\mathbf{V}(\Omega_{Y/k})$ over x. In general, $\mathbf{V}\Omega_{Y/Z}$ is called the tangent space (or bundle) of Y/Z.

Corollary 10 *Let X/k be a scheme locally of finite type, where k is algebraically closed. Then the dimension of m_x/m_x^2 is an uppersemicontinuous function on the set of closed points of X.*

Proof: This is because $\Omega_{X/k}$ is a quasi-coherent sheaf of finite type on X (hence coherent, since X is noetherian), and it follows from Nakayama's lemma that the dimension of $\Omega_{X/k}(x)$ is upper semicontinous. In fact: $\square$

Lemma 11 *Let X be a scheme and let E be a quasi-coherent sheaf of $\mathcal{O}_X$-modules which is locally finitely generated. Then the dimension of $E(x)$ is uppersemicontinuous on X. If E is locally free, it is in fact locally constant, and the converse holds if X is reduced.*

Proof: Let x be a point of X, and let $(e_1(x), \ldots e_d(x))$ be a basis for $E(x) := E_x/m_{X,x}E_x$. There exist an open affine neighborhood U of x and sections $(e_1, \ldots, e_d)$ of $E(U)$, such that the image of e_i in $E(x)$ is $e_i(x)$. Replace X by U and let $(\mathcal{O}_X)^n \to E$ be the corresponding map. It follows from Nakayama's lemma that the induced map on $(\mathcal{O}_{X,x})^n \to E_x$ is surjective, and hence that is is surjective in some neighborhood of x (again using the fact that E is finitely generated). Then $dimE(x') \leq dimE(x)$ for all x' in this neighborhood. Suppose the dimension is in fact constant. We may assume it is constant and that X is affine, say $X = \operatorname{Spec} A$, and E corresponds to a finitely generated A-module M. We have constructed a surjective map $A^n \to M$, where n is the dimension of $M \otimes k(x)$ for every $x \in \operatorname{Spec} A$. It follows that the map $k(x)^n \to M \otimes k(x)$ is bijective for every x. Let $K \subseteq A^n$ be the kernel of $A^n \to M$, and observe that any coordinate of any element of the kernel maps to zero in A_p/PA_P for every prime P. Since A is reduced, the intersection of all the primes is zero, so $K = 0$. $\square$

Example 12 The map $I/I^2 \to i^*\Omega_{X/Y}$ might not be injective, even if I is the maximal ideal corresponding to a closed point of a scheme X of finite type over $Y = \operatorname{Spec} k$. For example, let k be a field of characteristic p which is not perfect, with an element a which is not a pth power, let $X := \operatorname{Spec} k[X]$ and let I be the ideal generated by $f := X^p - a$. Since this polynomial is irreducible, I is maximal and corresponds to a closed point x. But $df = 0$, so the map in this case is zero.

2 Smooth, unramified, and étale morphisms

The following definition makes sense for any morphism of sheaves $X \to Y$, but we state it only for schemes.

Definition 13 *A morphism $f : X \to Y$ is*

1. formally smooth *if for every affine nth order thickening $S \to T$ over $Y m$ every $g \in X/Y(S)$, can be deformed to T;*

2. formally unramified *if for every $S \to T$ every g as above, has at most one deformation to T.*

3. formally étale *if it is both formally smooth and formally unramified.*

A morphism is smooth *if it is formally smoth and locally of finite presentation, is* unramified *if it is formally unramified and locally of finite type, and is étale if it is smooth and unramified.*

If f is formally étale, then the uniqueness of the deformations implies that the liftings on open covers agree on the overlaps and hence patch to a unique global lifting. Since every nth order thickening is a succession of first order thickenings, it suffice in the above definition just to consider first order thickenings. In this case, let $\mathcal{I}$ be the ideal sheaf of S in T. Then Def_g forms a pseudo-torsor under $\mathrm{Der}_{X/Y}(g_*\mathcal{I})$, and to say that f is formally smooth is to say that this pseudo-torsor is always in fact a torsor. To say that f is formally unramified is to say that the pseudo-torsor has at most one element.

Corollary 14 *A morphism $f: X \to Y$ is formally unramified if and only if $\Omega^1_{X/Y} = 0$.*

Proof: If $\Omega^1_{X/Y} = 0$, it follows from Theorem 6 that first order deformations are unique if they exist. Suppose conversely that f is unramified, let $\mathcal{E}$ be any quasi-coherent sheaf of X and let $X \to D_X(\mathcal{E})$ be the trivial extension of X by $\mathcal{E}$. Then the set of deformations of id_X to $D_X(\mathcal{E})$ is not empty and is a torsor under $\mathcal{H}om(\Omega^1_{X/Y}, \mathcal{E})$. It follows that this group is zero. Taking $\mathcal{E} = \Omega^1_{X/Y}$, we see that the latter must vanish. □

Theorem 15 *Let $f: X \to Y$ be a morphism of finite type. Then f is unramified if and only if its geometric fibers are finite, reduced, and discrete.*

Proof: Since f is of finite type, the sheaf $\Omega^1_{X/Y}$ is also of finite type. It vanishes if and only if for each $x \in X$, the stalk $\Omega^1_{X/Y}$ at x vanishes, and by Nakayama's lemma, this is true if and only if the fiber $\Omega^1_{X/Y}(x)$ vanishes, Let $y := f(x)$, which we identify with $\operatorname{Spec} k(y) \to Y$, and let $X_y := X \times_Y y$. Then if $p: X_y \to Y$ is the natural map, $p^*\Omega^1_{X/Y} \cong \Omega^1_{X_y/y}$. The point x of X defines a point x' of X_y with $p(x') = x$, and the above isomorphism identifies $\Omega^1_{X_y}(x')$ with $\Omega^1_{X/Y}(x)$. Thus we see that f is unramified if and only if every $X_y \to y$ is unramfied. Since $\overline{y} \to y$ is faithfully flat, $\Omega^1_{X_y/y} = 0$ if and only if $\Omega^1_{X_{\overline{y}}/\overline{y}}$ is unramified. □

Proposition 16 *If $f: X \to Y$ is smooth, then the sheaf $\Omega^1_{X/Y}$ is locally free.*

Theorem 17 *Let $f: X \to Y$ and $g: Y \to Z$ be morphisms of schemes, each locally of finite presentation, and let $h := g \circ f$.*

1. *Suppose that $f: X \to Y$ is a closed immersion defined by a sheaf of ideals $\mathcal{I}$. If $X \to Z$ is smooth, the map*

$$\overline{d}: \mathcal{I}/\mathcal{I}^2 \to f^*\Omega^1_{Y/Z}$$

   of Theorem 8 is injective and locally split. The converse is true provided that $Y \to Z$ is smooth.

2. *If f is smooth, the map*

$$f^*\Omega^1_{Y/Z} \to \Omega^1_{X/Z}$$

   of Theorem 8. is injective and locally split. The converse is true provided that h is smooth.

Proof: Suppose that $h: X \to Z$ is smooth. Let $j: Y_1 \to Y$ be the closed subscheme of Y defined by $\mathcal{I}^2$. Then $i: X \to Y_1$ is a first-order thickening. Assuming without loss of generality that X is affine, the smoothness of X/Z implies that there is a retraction $r: Y_1 \to X$, compatible with the given maps to Z. Let $\tilde{j} := f \circ r: Y_1 \to Y$, and note that $\tilde{j} \circ i = f \circ r \circ i = f = j \circ i$. Thus $j^\sharp - \tilde{j}^\sharp$ "is" a derivation $D : \mathcal{O}_Y \to \mathcal{I}/\mathcal{I}^2$. This derivation defines a map $\Omega^1_{Y/Z} \to \mathcal{I}/\mathcal{I}^2$ which gives the desired splitting.

For the converse, suppose that g is smooth, working locally, that

$$s: f^*\Omega^1_{Y/Z} \to \mathcal{I}/\mathcal{I}^2$$

splits $\overline{d}$. Let $i: S \to T$ be an affine first order thickening over Z, defined by a square zero ideal $\mathcal{J}$, and let $r: S \to X$ be a morphism (over Z). Since Y/Z is smooth, there exists a deformation s of $f \circ f$ to T. Then $s^\sharp: \mathcal{O}_Y \to s_*\mathcal{O}_T$ necessarily maps $\mathcal{I}$ to $r_*\mathcal{J}$, and since $\mathcal{J}^2 = 0$, factors through a map $\theta: \mathcal{I}/\mathcal{I}^2 \to s_*\mathcal{J}$. Then $\theta \circ s: f^*\Omega^1_{Y/Z} \to \mathcal{J}$, composed with the natural map $\Omega^1_{Y/Z} \to f_*f^*\Omega^1_{Y/Z}$ defines a derivation $D: \mathcal{O}_Y \to s_*\mathcal{J}$. Then $\tilde{s} := s - D$ is a map $T \to Y$, and in fact $\tilde{s}^\sharp$ kills $\mathcal{I}$ and hence $\tilde{s}$ factors through X.

We omit the proof of (2), which is quite similar. $\square$

Remark 18 In the situation of part (1) of the previous theorem, let x be a point of X, and suppose that $Y \to Z$ is smooth. The morphism $X \to Z$ is smooth in some neighborhood of x if and only if the map of $k(x)$-vector spaces:

$$d(x)\colon I(x) \to \Omega^1_{Y/Z}(x)$$

is injective. Smoothness implies that $\overline{d}$ is injective and locally split, and it follows immediately that $\overline{d}$ is injective. Conversely, if $d(x)$ is injective, choose a lift $f_1, \ldots, f_r$ to $\mathcal{I}_x$ of a basis for $\mathcal{I}(x)$. By Nakayama's lemma, these elements generate $\mathcal{I}_x$. (Recall that X is of finite presentation.) Then $(df_1(x), \ldots, df_r(x))$ remains linearly independing in $\Omega^1_{Y/Z}(x)$, and hence can be completeted to a basis$(df_1(x), \ldots, df_r(x), \omega_1(x), \omega_s(x))$, with $\omega_i \in \Omega^1_{Y/Z,x})$. Since $\Omega^1_{Y/Z,x}$ is free, it follows that $df_1, \ldots df_r, \omega_1, \omega_s)$ is a basis, and the desired splitting is easy to construct.

Corollary 19 *Let X/k be a scheme of finite type over a field k and let x be a k-rational point of X. Then X/k is smooth in some neighborhood of x if and only if X is regular at x.*

Proof: The question is local, so we may assume that X can be embededd in $Y := \mathbf{A}^n$. If $m_x\mathcal{O}_{Y,x}$ is the maximal ideal of the local ring $\mathcal{O}_{Y,x}$, the map $d(x)$ above identifies with the map $\mathcal{I}_x/m_x\mathcal{I}_x \to m_x/m_x^2$. Thus the map is injective if and only if $\mathcal{I}_x \cap m_x^2 \subseteq m_x\mathcal{I}_x$. Since Y is regular at x, this condition is equivalent to the regularity of X at x.

Let us recall the proof.

Since the closed points of X are dense and since localizations of regular local rings are regular, X is a regular scheme if and only if the local rings at the closed points of X are regular. Working locally, we may assume that X can be embedded as a closed subscheme of an affine space Y/k. Let x be a closed point of X. Corollary 18 tells us that X is smooth if and only if the map $\mathcal{I}(x) \to m_x/m_x^2$ is injective. This is equivalent to the regularity of X at x. Let us recall the proof. Let m_x be the maximal ideal of $\mathcal{O}_{Y,x}$ and let $\overline{m}_x = m_x/\mathcal{I}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$. We have exact sequences:

$$0 \to \mathcal{I}_x \cap m_x^2/\mathcal{I}_x m_x \to \mathcal{I}(x) \to \mathcal{I}_x/\mathcal{I}_x \cap m_x^2 \to 0$$

$$0 \to \mathcal{I}_x/\mathcal{I}_x \cap m_x^2 \to m_x/m_x^2 \to \overline{m}_x/\overline{m}_x^2 \to 0$$

where m_x is the maximal ideal of $\mathcal{O}_{Y,y}$ and $\overline{m}_x$ is the maximal ideal of $\mathcal{O}_{X,x}$. For simplicity of notation, rewrite these sequences as:

$$0 \to K(x) \to \mathcal{I}(x) \to \overline{\mathcal{I}}(x).$$

$$0 \to \overline{\mathcal{I}}(x) \to m_x/m_x^2 \to \overline{m}_x/\overline{m}_x^2 \to 0.$$

Since Y is regular, $\dim_x(Y) = \dim m_x/m_x^2$, and since $\mathcal{I}_x$ can be generated by $\dim \mathcal{I}(x)$ elements,

$$\begin{array}{rcl} \dim_x(X) & \geq & \dim_x(Y) - \dim \mathcal{I}(x) \\ & \geq & \dim(m_x/m_x^2) - \dim(K(x)) - \dim(\overline{\mathcal{I}}(x)) \\ & \geq & \dim(\overline{m}_x/\overline{m}_x^2) - \dim(K(x)) \end{array}$$

If $K(x) = 0$ we conclude that $\dim_x(X) \geq \dim(\overline{m}_x/\overline{m}_x^2)$, and it follows that in fact equality holds and X is regular at x. For the converse, suppose that X is regular and choose a a sequence of elements $f_1, \ldots, f_r$ of $\mathcal{I}_x$ lifting a basis for $\overline{\mathcal{I}}(x)$. Then $f_1(x) \cdots f_r(x) \in \mathcal{I}(x)$ are linearly independent in m_x/m_x^2, and hence can be completed to a basis for this vector space. Lifting the remaining elements to elements of m_x, we end up with a sequence $(f_1, \ldots, f_n)$ of elements of m_x such that $(f_1, \ldots, f_r)$ lie in $\mathcal{I}_x$ and such that $(f_1(x), \ldots, f_n(x)$ is a basis for the maximal m_x of x in $\mathcal{O}_{Y,x}$. Let J be the ideal $\mathcal{O}_{Y,y}$ generated by $(f_1, \ldots, f_r)$. Then it follows from the argument above that $X' := \operatorname{Spec} \mathcal{O}_{Y,y}/J$ is regular of dimension $\dim_x Y - \dim(\overline{\mathcal{I}}_x) = \dim \overline{m}_x/ovm_x^2$. If X is regular at x, then this is also the dimension of X. But $X \subseteq X'$ and since X' is regular, it is irreducible, and it follows that X and X' coincide at x. Then $J = \mathcal{I}_x$ and it follows that $K(x) = 0$. □

Theorem 20 *Let $f \colon X \to Z$ be a morphism locally of finite presentation. Assume that X and Y are locally noetherian. Then f is smooth if and only if it is flat and its geometric fibers are regular.*

Proof: Assume that f is smooth. Let x be a point of X, let $z := f(x)$, and let $\overline{z}$ be the spectrum of algebraically closed field endowed with a map to z. Then $X_{\overline{z}}$ is smooth, and hence by Corollary 19 it is regular. Our task is to prove that f is flat. We use the following techinique from commutative algebra.

Lemma 21 *Let $R \to B$ be a local homomorphism of noetherian local rings.*

1. *B is flat over R if and only if $Tor_1^R(B, k_R) = 0$.*
2. *Suppose that B is flat over R and that A is the quotient of B by the ideal I generated by an element b of m_B. If the image $\overline{b}$ of b in $B/m_R B$ is a nonzero divisor, then A is also flat over R.*

Proof: Statement (1) is the famous "local criterion of flatness," and we use it to prove (2). Since B is R flat, $Tor_1^R(B, k_R) = 0$, so the top row of the diagram below is exact:

$$\begin{tikzcd}
0 \arrow[r] & Tor_1^R(A, k_R) \arrow[r] & I \otimes_R k_R \arrow[r, "\beta"] & B \otimes_R k_R \\
& & B \otimes_R k_R \arrow[u, "\overline{\pi}"] \arrow[ur, "\cdot\overline{b}"'] &
\end{tikzcd}$$

By assumption $\cdot\overline{b}$ is injective, and it follows that $\overline{\pi}$ is also injective. The map $\overline{\pi}$ is obtained from the surjective map $B \to I$ and hence is also surjective, hence bijective. Then it follows that β is injective and then that $Tor_1^R(A, k_R) = 0$. $\square$

We return to the proof of the theorem. Since $X_{\overline{z}} \to X_z$ is flat, it follows that X_z is regular, and in particular is an integral domain. Working locally, we may assume that there exists a closed immersion $i: X \to Y$, where Y is both flat and smooth over Z—for example affine n-space over Z. Let $R := \mathcal{O}_{Z,z}, B = \mathcal{O}_{Y,x}$, and $A := \mathcal{O}_{X,x}$. Let I be the kernel of the surjection $B \to I$. By Theorem 17, the map $\overline{d}: I(x) \to \Omega^1_{Y/Z}(x)$ is injective. Choose a sequence of generators $(b_1, \ldots, b_r)$ for I such that $(f_1(x), \ldots, f_r(x))$ is a basis for $I(x)$ Theorem 17 implies that for every i, the subscheme X'_i of Y defined by $(b_1, \ldots, b_i)$ is again smooth over Z. We prove that it is flat by induction on i. This is true by assumption if $i = 0$, and the general induction step will follow from the case $i = 1$. Since Y/R is smooth, its fibers are regular, and hence $B \otimes_R k$ is an integral domain. Since the image $\overline{b}$ of b_1 in $B \otimes_R k$ is not zero, $\overline{b}$ is a nonzero divisor, and since B is flat, the lemma implies that A is also flat. This completes the proof.

For the converse, suppose that $X \to Z$ is flat and that its geometric fibers are regular. We claim that $X \to Z$ is smooth. Again we work locally in a neighborhood of a point x, so we can assume that there is a closed immersion $i: X \to Y$ where Y is smooth over Z (e.g. affine space). Suppose $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$, with $A = B/I$. If $\overline{z}$ is a geometric point lying over the image $z \in Z$ of X, then by Corollary 19 we know that $X \times_z \overline{z}$ is smooth over $\overline{z}$. Let $\overline{B} := B \otimes_R k$ and $\overline{A} := A \otimes_R k$ and let $\overline{I}$ be the kernel of the map $B \otimes_R k \to A \otimes_R k$. Since A/R is flat, in fact $\overline{I} = I \otimes_R k_R$. Then

$$I(x) := I/ot_B k(x) = I \otimes_B k_R \otimes_{\overline{B}} k(x) \cong \overline{I} \otimes_{\overline{B}} k(x),$$

and $\overline{d}\colon I(x) \to \Omega^1_{B/R}(x)$ identifies with the map $\overline{I}(x) \to \Omega^1_{\overline{B}/k_R}(x)$. tensoring over k_R with $k(z)$, we get the corresponding map for the geometric fibers $X_{\overline{z}} \to Y_{\overline{z}}$, which is injective. $\square$