Cohomology and Base Change

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $T\colon \mathcal{A} \to \mathcal{B}$ and additive functor. We say T is *half-exact* if whenever $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, the sequence $T(M') \to T(M) \to T(M'')$ is exact.

Lemma 1 *Let $\eta\colon T' \to T$ be a morphism of half exact functors, and for each M, let $T''(M)$ be the cokernel of η_M. Then T'' is half-exact if T' is right exact.*

Proof: If $0 \to M' \to M \to M'' \to 0$ is an exact sequece, we get a commutative diagram:

$$\begin{array}{ccccccc}
T'(M') & \longrightarrow & T'(M) & \longrightarrow & T'(M'') & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
T(M') & \longrightarrow & T(M) & \longrightarrow & T(M'') & & \\
\downarrow & & \downarrow & & \downarrow & & \\
T''(M') & \longrightarrow & T''(M) & \longrightarrow & T''(M'') & &
\end{array}$$

The rows and columns are exact and the bottom vertical arrows are surjective. A diagram chase shows that the bottom row is exact. $\square$

The following useful lemma is not especially well known.

Theorem 2 (Nakayama's lemma for half-exact functors) *Let $A \to B$ be a local homomorism of noetherian local rings and T a half-exact functor A-linear functor from the category of finitely generated A-modules to the category of finitely generated B-modules. Let k be the residue field of A. Then if $T(k) = 0$, in fact $T(M) = 0$ for all M.*

Proof: Our hypothesis is that $T(k) = 0$, and we want to conclude that $T(M) = 0$ for every finitely generated A-module M. Consider the family $\mathcal{F}$ of submodules M' of M such that $T(M/M') \neq 0$. Our claim is that the 0 submodule does not belong to $\mathcal{F}$, and so of course it will suffice to prove that $\mathcal{F}$ is empty. Assuming otherwise, we see from the fact that M is noetherian that $\mathcal{F}$ has a maximal element M'. Let $M'' := M/M'$. Then $T(M'') \neq 0$, but $T(M''') = 0$ for every nontrivial quotient M''' of M''. We shall see that this leads to a contradiction. To simply the notation, we replace M by M''. Thus it suffices to prove that $T(M) = 0$ under the assumption that $T(M'') = 0$ for every proper quotient of M.

Let I be the annihilator of M and let m denote the maximal ideal of A. If $I = m$, then M is isomorphic to a direct sum of a (finite number of) copies of k. By assumption, $T(k) = 0$, hence $T(M) = 0$, a contradiction. So there exists an element a of m such that $a \notin I$. Let M' be the kernel of multiplication of a on M, so that there are exact sequences:

$$\begin{array}{rcl} 0 \to M' \to & M & \to aM \to 0 \\ 0 \to aM \to & M & \to M/aM \to 0, \end{array}$$

and hence also exact sequences:

$$\begin{array}{rcl} T(M') \to & T(M) & \to T(aM) \\ T(aM) \to & T(M) & \to T(M/aM). \end{array}$$

Then $aM \neq 0$, since a does not belong to the annihilator of M, and hence M/aM is a proper quotient of M, and hence $T(M/aM) = 0$.

Case 1: If $M' \neq 0$, then the first sequence above shows that aM is also a proper quotient of M, and hence also $T(aM) = 0$, and then the last sequence implies that $T(M) = 0$.

Case 2: If $M' = 0$, we find exact sequences

$$\begin{array}{rcl} 0 \to M \xrightarrow{a} & M & \longrightarrow M/aM \to 0 \\ T(M) \xrightarrow{a} & T(M) & \longrightarrow T(M/aM) \end{array}$$

However we still know that $T(M/aM) = 0$, hence multiplication by a on $T(M)$ is surjective. But $T(M)$ is a finitely generated B-module and multiplication by a on this module is the same as multiplication by $\theta(a)$. Since $\theta(a)$ belongs to the maximal ideal of B, Nakayama's lemma implies that $T(M) = 0$, as required. □

Here are two important appications to flatness.

Theorem 3 (local criterion for flatness) *Let $\theta: A \to B$ be a local homomorphism of noetherian local rings and let N be a finitely generated B-module. Then N is flat as an A-module if and only if $Tor_1^A(N, A/\mathrm{m}) = 0$.*

Proof: To prove that N is flat as an A-module, it suffices to prove that $Tor_1^A(N, M) = 0$ for all finitely generated A-modules M. The functor $M \mapsto Tor_1^A(N, M)$ is half-exact, A-linear, and takes finitely generated A-modules to finitely generated B-modules. Thus the result follows from Nakayama's lemma for the half exact functor $Tor_1^A(N, \ )$. □

Theorem 4 (Criterion of Flatness along the Fiber) *Let $R \to A$ and $A \to B$ be local homomorphisms of noetherian local rings, let k be the residue field of R, and let N be a finitely generated B-module. If N is flat over R and $N \otimes_R k$ is flat over $A \otimes_R k$, then N is flat over A.*

Proof: We will need the following:

Lemma 5 *Let A be a ring, let I be an ideal of A, and let M be an A-module. Suppose that M/IM is flat as an A/I-module and also that $Tor_1^A(A/I, M) = 0$. Then $Tor_1^A(A/J, M) = 0$ for every ideal J containing I.*

Proof: Let $0 \to K \to F \to M \to 0$ be an exact sequence of A-modules, with F free. Since $\mathrm{Tor}_1^A(A/I, M) = 0$, the sequence

$$(*) \quad 0 \to K/IK \to F/IF \to M/IM \to 0$$

is an exact sequence of A/I-modules. Since M/IM is flat over A/I, the sequence remains exact if we tensor over A/I with A/J, so the sequence

$$0 \to K/JK \to F/JF \to M/JM \to 0$$

is still exact. However, F is free over A, and we also have an exact sequence

$$0 \to Tor_1^A(M, A/J) \to K/JK \to F/JF \to M/JM \to 0$$

Hence $\mathrm{Tor}_1^A(A/J, M) = 0$. □

Now to prove the theorem, let m_R be the maximal ideal of R and let $I := \mathrm{m}_R A$. Then we have a surjective map $\mathrm{m}_R \otimes_R A \to I$, and hence also a surjective map:

$$\mathrm{m}_R \otimes_R A \otimes_A N \to I \otimes_A N.$$

Since $\mathrm{m}_R \otimes_R A \otimes_A N \cong \mathrm{m}_R \otimes_R N$, we find maps

$$\mathrm{m}_R \otimes_R N \xrightarrow{f} I \otimes_A N \xrightarrow{g} N$$

We have just seen that f is surjective. On the other hand, the kernel of $g \circ f$ is $\mathrm{Tor}_R^1(k, N) = 0$, so $g \circ f$ is injective. Then it follows that f is also injective, hence an isomorphism, and hence that g is injective. The kernel of g is $\mathrm{Tor}_1^A(A/I, N)$, so this also vanishes. Furthermore, $N/IN \cong N \otimes_A (A/I) \cong N \otimes_A A \otimes_R k \cong N \otimes_R k$, and since N is flat over R, $N \otimes_R k$ is flat over $A \otimes_R k \cong A/I$. By the lemma, it follows that $\mathrm{Tor}_1^A(A/J, N) = 0$ for every ideal J containing I and in particular for J equal to the maximal ideal of A. By the local criterion for flatness, this implies that N is flat over A. □

Proposition 6 *Let A be a ring, and let T be an A-linear homomorphism from the category of A-modules to itself. Then there is a natural transformation $\eta\colon T(A)\otimes \ \to T$. This functor is an isomorphism if and only if T is right exact and commutes with direct limits.*

Proof: An element x of M defines a homomorphism $\theta_x\colon A \to M$ and hence a homomorphism $T(\theta_x)\colon T(A) \to T(M)$. Define $T(A) \times M \to T(M)$ by $(t, x) \mapsto \theta_x(t)$. This map is bilinear and hence defines a natural homomorphism

$$\eta_M\colon T(A) \otimes M \to T(M).$$

Since T is additive, it commutes with finite direct sums, and hence η_M is an isomorphism if M is free and finitely generated. Assume that T is right exact and commuts with direct limits. For any M, there is an exact sequence:

$$0 \to K \to E \to M \to 0,$$

with E free and finitely generated, and hence a commutative diagram:

$$\begin{array}{ccccccc}
T(A)\otimes K & \longrightarrow & T(A)\otimes E & \longrightarrow & T(A)\otimes M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
T(K) & \longrightarrow & T(E) & \longrightarrow & T(M) & \longrightarrow & 0
\end{array}$$

The middle vertical arrow is an isomorphism, and it follows that the right vertical arrow is surjective. Since A is noetherian, K is finitely generated, and we can conclude that the left vertical arrow is surjective. Then the right vertical arrow is an isomorphism. This proves that η_M is an isomorphism if M is finitely generated, and the same is true for all M since both sides commute with direct limits. The converse is obvious, since tensor products are right exact and commute with direct limits. □

Corollary 7 *Let $A \to B$ be a local homomorphism of noetherian local rings and let T be a half exact and linear functor from the category of noetherian A-modules to itself. Assume that T takes finitely generated modules to finitely generated modules and commutes with direct limits.*

1. *If $T(k) = 0$, then $T(M) = 0$ for all M.*
2. *If $T(A) \to T(k)$ is surjective, then T is right exact and the natural transformation $T(A)\otimes \;\; \to T$ is an isomorphism.*

Proof: Statement (1) follows from Nakayama's lemma for half-exact functors. To prove (2), let $T'(M) := T(A) \otimes_A M$ and let $T''(M)$ be the cokernel of the map $\eta_M T'(M) \to T(M)$ defined above. The functor T'' takes finitely generated modules to finitely generated modules and is is half exact by Lemma 1. By Nakayama's lemma for half-exact functors, $T'' = 0$. Then η_M is surjective for all M. It follows easily that T is right exact, and hence that η_M is an isomorphism by Proposition 6. Note that for statement (2), we just need to assume that $T(A)$ is finitely generated. □

Theorem 8 (cohomology and base change) *Let X/S be a proper scheme, where $S = \operatorname{Spec} A$ and A is a noetherian local ring, and let E be a coherent sheaf on X which is flat over S. If M is an A-module, let $E \otimes M$ denote the quasi-coherent sheaf on X obtained by tensoring the pullback of $\tilde{M}$ to X with E. In particular, if M is the residue field of A, $E \otimes M$ identifies with restriction of E to the closed fiber X_k over k. Then the following statement hold.*

1. *If $H^q(X_k, E_k) = 0$, then $H^q(X, E \otimes M) = 0$ for all M.*
2. *If $H^q(X, E) \otimes_A k \to H^q(X_k, E_k)$ is surjective, then it is an isomorphism, and in fact $H^q(X, E) \otimes_A M \to H^q(X, E \otimes M)$ is an isomorphism for all M.*

3. *Suppose that* $H^q(X,E)\otimes_A k \to H^q(X_k,E_k)$ *is surjective. Then* $H^q(X,E)$ *is flat if and only if* $H^{q-1}(X,E)\otimes k \to H^{q-1}(X_k,E_k)$ *is surjective.*

Proof: For each q, let T^q be the functor taking an A-module M to $H^q(X, E\otimes M)$. Since E is flat, an exact sequence $0 \to M' \to M \to M'' \to 0$ yields an exact sequence $0 \to E\otimes M' \to E\otimes M \to E\otimes M'' \to 0$, and so the functors $T^{\cdot}$ fit into a cohomological δ-functor. In particiular, each T^q is half-exact. Furthermore it takes finitely generated modules to finitely generated modules because X/A is proper, and it commutes with direct limits by a Cech calculation, or using the fact that X is noetherian. Thus statements (1) and (2) follow from the corresponding statements of Corollary 7. To prove (3), note that since $T^{\cdot}$ forms a cohomological δ-functor, T^q is left exact if and only if T^{q-1} is right exact. If $H^q(X,E)\otimes k \to H^q(X_k,E_k)$ is surjective, then $T^q(M) = H^q(E\otimes M) \cong H^q(E)\otimes_A M$ for all M. Thus $H^q(E)$ is flat if and only if T^q is left exact, which is the case if and only if T^{q-1} is right exact, equivalently, if and only if $H^{q-1}(X,E)\otimes k \to H^{q-1}(X_k,E_k)$ is surjective. $\square$