Homework Assignment #8:

Due April 19

For some of the problems, you will need to use the technique of cohomology and base change. A simplified treatment of the essential facts can be found in the Notes section of the course web page.

1. Let X/k be a smooth proper curve over an algebraically closed field k, and let $\mathcal{L}$ be an invertible sheaf on X. Suppose that $V := \Gamma(X, \mathcal{L})$ has dimension at least two and generates $\mathcal{L}$. Show that V contains a two-dimensional subspace which generates $\mathcal{L}$.

2. Let X/k be a smooth proper curve over an algebraically closed field k, of genus g. Prove that there exists a morphism $X \to \mathbf{P}^1$ of degree less than or equal to $g+1$.

3. Let X/S be a proper and flat morphism, where $S = \operatorname{Spec} A$ and A is a noetherian ring. Suppose that all the geometric fibers of X/A are nonempty, connected and reduced. Then the natural map $A \to \Gamma(X, \mathcal{O}_X)$ is an isomorphism.

4. Let $f\colon X \to S$ be a smooth proper morphism such that for every $s \in S$, the fiber $X(s)$ is a geometrically connected curve of genus one. If $\mathcal{L}$ is a an invertible sheaf on such an X, we let $\deg(\mathcal{L})$ be the function which takes a point s to the degree of the restriction of $\mathcal{L}$ to the fiber $X(s)$. This function is always locally constant, we will typically just assume it is constant. Assume for simplicity here that all schemes are noetherian.

   (a) Prove that if $\deg(\mathcal{L}) > 0$, then $R^1 f_* \mathcal{L} = 0$, that $f_*\mathcal{L}$ is a locally free sheaf of rank $\deg(\mathcal{L})$, and that for any $g\colon S' \to S$, the map $g^* f_* \mathcal{L} \to f'_* \mathcal{L}'$ is an isomorphism, where $\mathcal{L}'$ is the inverse image of $\mathcal{L}$ on $X' := X \times_S S'$. It suffice to do the case in which S is affine.

(b) Let $D_1(S) := Div_1(X/S)$ denote the set of isomorphism classes of pairs $(\mathcal{L}, s)$, where $\mathcal{L}$ is an invertible sheaf on X of degree one and s is an isomorphism $\mathcal{O}_S \to f_*\mathcal{L}$. Prove that if $g\colon S' \to S$ is any map, then $(\mathcal{L}', g^*s) \in Div_1(X'/S')$, so that D_1 becomes a functor on the category of schemes over S.

(c) If T is any scheme, let $Pic(T)$ denote the set of isomorphism classes of invertible sheaves on T. If $(X/S, \sigma)$ is an elliptic curve as above, let $Pic(X/S)$ denote the cokernel of the map $Pic(S) \to Pic(X)$. Note that deg factors through $Pic(X/S)$, and let $Pic_d(X/S)$ denote the set of elements of degree d. This set is a torsor under the group $Pic_0(X/S)$. If $S' \to S$ is a morphism, base change defines a natural map $Pic(X/S) \to Pic(X'/S')$, compatible with *deg*. Let $P_1(S)$ denote $Pic_1(X/S)$, a functor in S. Define a map $D_1 \to P_1$ by sending the isomorphism class of a pair $(\mathcal{L}, s)$ to the image of $\mathcal{L}$ in $P_1(S)$. Show that this map is a bijection. (Hint: If ℓ is an element of $P_1(S)$, choose some invertible sheaf $\mathcal{L}$ of degree one whose isomorphism class in $Pic(X)$ lies in ℓ. Let $\mathcal{M} := f_*\mathcal{L}$ and let $\mathcal{L}' := \mathcal{H}om(f^*\mathcal{M}, \mathcal{L})$. Prove that the natural map $f^*\mathcal{M} \to \mathcal{L}$ defines a basis s for $f_*\mathcal{L}'$, and the class of $(\mathcal{L}', s)$ in $P_1(S)$ is ℓ.)

(d) Let X/S be the functor taking an S-scheme S' to the set of S-morphisms $S' \to X$, or equivalently, the set of sections τ of X'/S'. If $\tau \in X/S(S)$, the image of τ is a closed subscheme of X, and its ideal sheaf $\mathcal{I}_\tau$ is invertible. Prove that the inclusion $\mathcal{I}_\tau \to \mathcal{O}_X$ defines a basis for $f_*\mathcal{L}_\tau$, where $\mathcal{L}_\tau := \mathcal{H}om(\mathcal{I}_\tau, \mathcal{O}_X)$. This defines a morphism $X/S \to D_1$. Prove that this morphism an isomorphism. (Hint: If $(\mathcal{L}, s) \in D_1(S)$, prove that the image of the map $\mathcal{O}_X \to \mathcal{L}$ defined by s is $\mathcal{I}\mathcal{L}$, where $\mathcal{I}$ is an invertible sheaf of ideals defining a section τ of X/S.)

(e) Now suppose that σ is fixed section of X/S, which you can use to identify the functor P_1 with the functor P_0 and hence the functor X/S with the functor P_0. Deduce that X/S has a group scheme structure with σ as the idenity section.