Homework Assignment #7:

Due April 7

Let $\theta\colon R \to A$ be a homomorphism of commutative rings. Let $\omega_{A/R} := \mathrm{Hom}_R(A, R)$, viewed as an A-module via A, and let $t_{A/R}\colon \omega_{A/R} \to R$ be the homomorphism sending h to $h(1)$.

1. Show that the pair $(t_{A/R}, \omega_{A/R})$ represents the functor taking an A-module M to the R-module $\mathrm{Hom}_R(\theta_*(M), R)$.
2. If $A = R[X]/(f)$, where f is a monic polynomial, show that $\omega_{A/R}$ is a free A-module of rank one.
3. If $A = R[x, y]/(x^2, xy, y^2)$, show that $\omega_{A/R}$ is not free as an A-module.
4. If $\theta\colon R \to A$ is finite and flat, then for each $a \in A$, let $\mathrm{tr}_{A/R}(a)$ be the trace of the endomorphism of A defined by multiplication by a. Then $\mathrm{tr}_{A/R}$ is an R-linear map $A \to R$ and hence corresponds to an R-linear map $\mathrm{Tr}_{A/R}\colon A \to \omega_{A/R}$. This map is an isomorphism if and only if A/R is smooth (hence étale). Prove this fact by hand if $A = R[X]/(f)$ where f is monic of degree 2.
5. Hartshorne III, 7.2
6. Hartshorne III, 7.3