Homework Assignment #3:

Due February 22

1. Let R be a ring and let E be an R-module. Recall that the functor $\mathbf{V}E$ taking an R-algebra A to the set $\mathrm{Hom}_R(E, A)$ is representable by the R-algebra $S^{\cdot}E$ together with the universal element $v\colon E \to S^{\cdot}E$ (the inclusion of E into the degree one component of $S^{\cdot}E$). This functor has its values in the category of sets, but in fact it factors naturally through the category of abelian groups. Identifying the scheme $\mathrm{Spec}\, S^{\cdot}E$ with the functor $\mathbf{V}E$, and using Yoneda, we find a natural group scheme structure $\mathbf{V}E \times_R \mathbf{V}E \to \mathbf{V}E$. Recall that there is a unique derivation

$$d\colon S^{\cdot}E \to S^{\cdot}E \otimes E$$

such that $d \circ v(e) = 1 \otimes e$ for all $e \in E$ and that this derivation is universal, so that $\Omega^1_{S^{\cdot}E/R} = S^{\cdot}E \otimes E$. If I is a square zero ideal in A, how that the action of $\mathbf{V}E(A)$ on itself defined by the group structure is compatible with the the action of $\mathrm{Der}_{S^{\cdot}(E)/R}(I)$. (The only difficulty here is figuring out how to say this, and perhaps the signs.)

2. Let R be a ring and let E be an R-module. Recall that $\mathbf{P}E$ is the functor taking to the set of hyperlanes in $A \otimes_R E$, equivalently, the set of isomorphism classes of invertible quotients $\ell\colon A \otimes_R E \to L$ of $A \otimes_R E$. Recall also that $\mathbf{P}E$ is covered by affine open subfunctors $D^+(e)$, where $D^+(e)$ is the set of isomorphism classes of invertible quotients ℓ such that $\ell(1 \otimes e)$ generates L; equivalently, the set of R-linear maps $v\colon E \to A$ such that $v(e) = 1$. Let E' be the quotient of E by the submodule of E generated by e. We have a closed immersion $\mathbf{P}E' \to \mathbf{P}E$. Show that $D^+(e)$ is the complement of this closed immersion. Show that the group valued functor $\mathbf{V}E'$ acts on the functor $D^+(e)$ and that this action makes $D^+(e)$ is a pseudo-torsor. (This generalizes the fact that $\mathbf{P}^n \setminus \mathbf{P}^{n-1} \cong \mathbf{A}^{n-1}$.)

3. Taking account the degrees, the universal derivation in Problem 1 defines a map
$$d\colon S^{\cdot}E \to S^{\cdot}E(-1) \otimes E$$
and multiplication defines a map $S^{\cdot}E(-1) \otimes E \to S^{\cdot}E$. Show that the composition of these two maps is just multiplication by m in degree m. Show that the maps and formula remain valid after localization by any homogeneous element g of $S^{\cdot}E$. Deduce that on $\mathbf{P}E$, there are maps
$$d_n\colon \mathcal{O}_{\mathbf{P}E}(n) \to \mathcal{O}_{\mathbf{P}E}(n-1) \otimes E$$
and that the map d_0 factors through a map $d\colon \mathcal{O}_{\mathbf{P}E} \to \mathcal{H}(-1)$, where $\mathcal{H} \subseteq \mathcal{O}_{\mathbf{P}E} \otimes E$ is the universal hyperplane.

4. Show that the map $d\colon \mathcal{O}_{\mathbf{P}E} \to \mathcal{H}(-1)$ constructed above is the universal derivation and defines a canonical isomorphism: $\Omega^1_{\mathbf{P}E/R} \to \mathcal{H}(-1)$.

5. Let k be a field of characteristic not equal to 3, let $R := k[t]$, and let $f := X^3 + Y^3 + Z^3 - 3tXYZ \in R[X,Y,Z]$. The ideal of $R[X,Y,Z]$ generated by f definex a closed subscheme X of $\mathbf{P}^3_R$. At which points of X does the morphism $X \to \operatorname{Spec} R$ fail to be smooth? Answer the analogous question for $g := t(X^3 + Y^3 + Z^3) - 3XYZ \in R[X,Y,Z]$.