Homework Assignment #2:

Due February 15

1. Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Assume that $\mathcal{L}$ is ample, in the following sense. For every quasi-coherent sheaf of ideals $\mathcal{I}$ on X and every $x \in X$ at which $\mathcal{I}_x = \mathcal{O}_{X,x}$, there exists an $n > 0$ and a section s of $\Gamma(X, \mathcal{I} \otimes \mathcal{L}^n)$ such that $s(x) \neq 0$. Prove that X is separated. Conclude that the quasi-separation hypothesis on the theorem I proved in class is superflous.
   Remark: For each s constructed as above, we have an affine open set $X_s := \{x \in X : s(x) \neq 0\}$, and the set of these forms a cover of X. Thus it is enough to prove that the the intersection of the diagonal of X with $X_s \times X_t$ is closed in $X_s \times X_t$, *i.e.*, that the natural map $X_s \cap X_t \to X_s \times X_t$ is a closed immersion. Note that $X_s \cap X_t = X_{st}$, so are reduced to proving that
   $$\Gamma(X_s, \mathcal{O}_X \otimes \Gamma(X_t, \mathcal{O}_X) \to \Gamma(X_{st}, \mathcal{O}_X)$$
   is surjective. Now we know that:
   $$X_s = \operatorname{Spec} \Gamma(X_s, \mathcal{O}_{X_s}) = \varinjlim(\Gamma(X, \mathcal{L}^n), \cdot s)$$
   and similarly for X_t and X_{st}. Then we need to prove that the natural map
   $$\varinjlim(\Gamma(X, \mathcal{L}^n), \cdot s) \otimes \varinjlim(\Gamma(X, \mathcal{L}^n), \cdot t) \to \varinjlim(\Gamma(X, \mathcal{L}^n), \cdot st)$$
   is surjective. This is easy.

2. Prove that, with the definition above, an invertible sheaf $\mathcal{L}$ on X is ample if and only if its restriction to X_{red} is ample.
   Remark: This is probably false as stated, without a noetherian hypothesis. If X is noetherian, the ideal $\mathcal{I}$ of X_{red} in X is nilpotent, and one can reduce to the case in which $\mathcal{I}^2 = 0$ by induction. Let

i be the inclusion of X_{red} in X. If $i^*\mathcal{L}$ is ample, to show that $\mathcal{L}$ is ample we must show that $\mathcal{L}^n$ has "enough" sections for $n >> 0$, and the difficulty is that a section of $i^*(\mathcal{L}^n)$ does not automatically lift to a section of $\mathcal{L}$. The obsruction is an element of $H^1(X, \mathcal{I}\mathcal{L}^n)$ (the class of a torsor). Since $\mathcal{I}^2 = 0$, the sheaf $\mathcal{I}\mathcal{L}^n$ lives on X_{red}, where $\mathcal{L}$ is ample, and it can be shown that the torsor can be trivialized after increasing n. This will be easier to do a bit later.

3. Let $i: Y \to U$ be an closed immersion and let $j: U \to X$ be an open immersion. Prove that if $U \to X$ is quasi-compact, then there also exist an open immersion $j': Y \to Z$ and a closed immersion $i': Z \to X$ such that $j \circ i = i' \circ j'$. Can you find an example showing that the quasi-compact hypothesis is not superflous? (I haven't yet.)
   Remark: E. Chen found the following reference:
   http://stacks.math.columbia.edu/tag/01QW,

4. Let us allow ourselves to use the following fact: If $f: X \to Y$ is a proper morphism of noetherian schemes, then $f_*(\mathcal{O}_X)$ is a coherent sheaf of $\mathcal{O}_Y$-algebras. Prove:

   (a) If X/k is a proper scheme over an algebraically closed field and $\mathcal{O}_X$ is ample, then X consists of a finite set of points (not necessarily reduced).

   (b) Let E be a vector space over k, let $f: X \to \mathbf{P}E$ be a proper morphism, with X/k proper, and let $\mathcal{L} := f^*(\mathcal{O}_{\mathbf{P}E}(1))$. Let Z be a connected closed subscheme of X. Prove that $f(Z)$ is a single point iff the restriction of $\mathcal{L}$ to Z is isomorphic to $\mathcal{O}_Z$.