Homework Assignment #2:

Due February 15

1. Let X be a scheme and $\mathcal{L}$ an invertible sheaf on X. Assume that $\mathcal{L}$ is ample, in the following sense. For every quasi-coherent sheaf of ideals $\mathcal{I}$ on X and every $x \in X$ at which $\mathcal{I}_x = \mathcal{O}_{X,x}$, there exists an $n > 0$ and a section s of $\Gamma(X, \mathcal{I} \otimes \mathcal{L}^n)$ such that $s(x) \neq 0$. Prove that X is separated. Conclude that the quasi-separation hypothesis on the theorem I proved in class is superflous.

2. Prove that, with the definition above, a sheaf on X is ample if and only if its restrction to X_{red} is ample.

3. Let $i: Y \to U$ be an closed immersion and let $j: U \to X$ be an open immersion. Prove that if $U \to X$ is quasi-compact, then there also exist an open immersion $j': Y \to Z$ and a closed immersion $i': Z \to X$ such that $j \circ i = i' \circ j'$. Can you find an example showing that the quasi-compact hypothesis is not superflous? (I haven't yet.)

4. Let us allow ourselves to use the following fact: If $f: X \to Y$ is a proper morphism of noetherian schemes, then $f_*(\mathcal{O}_X)$ is a coherent sheaf of $\mathcal{O}_Y$-algebras. Prove:

   (a) If X/k is a proper scheme over an algebraically closed field and $\mathcal{O}_X$ is ample, then X consists of a finite set of points (not necessarily reduced).

   (b) Let E be a vector space over k, let $f: X \to \mathbf{P}E$ be a morphism, with X/k proper, and let $\mathcal{L} := f^*(\mathcal{O}_{\mathbf{P}E}(1))$. Let Z be a connected closed subscheme of X. Prove that $f(Z)$ is a single point iff the restriction of $\mathcal{L}$ to Z is isomorphic to $\mathcal{O}_Z$.