Regular local rings I

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Let $\mathcal{O}$ be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $e(\mathcal{O})$ denote the dimension of the k-vector space $\mathfrak{m}^2$. Let

$$\mathrm{Gr}_{\mathfrak{m}}(\mathcal{O}) := \oplus \mathfrak{m}^i \mathfrak{m}^{i+1},$$

viewed as a graded k-algebra.

Recall that Krull's theorem implies that $\cap \mathfrak{m}^i = \{0\}$. Hence if $a \in \mathcal{O}$ is nonzero, there exists a natural number ν such that $a \in \mathfrak{m}^\nu \setminus \mathfrak{m}^{\nu+1}$. We write $\nu(a)$ when there is room, and we write $In(a)$ for the image of a in $\mathfrak{m}^\nu/\mathfrak{m}^{\nu+1}$. Note that if $a, b \in \mathcal{O}$, then $\nu(ab) \geq \nu(a) + \nu(b)$. If equality holds, then $In(ab) = In(a)In(b)$, and this is true if and only if $In(a)In(b)$ is not zero. Furthermore, $\nu(a+b) \geq \min\{\nu(a), \nu(b)\}$.

If I is an ideal in $\mathcal{O}$, then for each integer ν, the image of $I \cap \mathfrak{m}^\nu \to \mathrm{Gr}^\nu_{\mathfrak{m}} \mathcal{O}$ is the set of initial forms of degree ν of elements of I. Summing over all ν, we get a subset $In(I)$ of $\mathrm{Gr}_{\mathfrak{m}}(\mathcal{O})$. The exact sequence

$$0 \to In(I) \to \mathrm{Gr}_{\mathfrak{m}}(\mathcal{O}) \to \mathrm{Gr}_{\mathfrak{m}}(\mathcal{O}/I) \to 0$$

shows that $In(I)$ is in fact an ideal of $\mathrm{Gr}_{\mathfrak{m}}(\mathcal{O})$.

Note: If $\mathrm{Gr}_{\mathfrak{m}}(\mathcal{O})$ is a domain and I is principally generated by f, then $In(I)$ is principally generated by $In(f)$. This is because every element of I is of the form fg for some $g \in \mathcal{O}$, and $In(fg) = In(f)In(g)$. As a consequence, we see that the following holds.

Proposition 0.1 *If $\mathrm{Gr}_{\mathfrak{m}}(\mathcal{O})$ is an integral domain, then $\mathcal{O}$ is an integral domain. Furthermore, in this case $\nu(ab) = \nu(a)+\nu(b)$ and $In(ab) = In(a)In(b)$ for any pair of nonzero elements of $\mathcal{O}$.*

The map $\mathfrak{m}/\mathfrak{m}^2 \to \mathrm{Gr}_{\mathfrak{m}}(\mathcal{O})$ extends uniquely to a homomorphism of graded k-algebras:

$$\sigma\colon S^{\cdot}(\mathfrak{m}/\mathfrak{m}^2) \to \mathrm{Gr}_{\mathfrak{m}}(\mathcal{O}).$$

Theorem 0.2 *The following are equivalent.*

1. *There is an $\mathcal{O}$-regular sequence which generates $\mathfrak{m}$.*
2. $\dim(\mathcal{O}) = e(\mathcal{O})$.
3. *The map σ above is an isomorphism.*

Proof: If $(x_1, \ldots, x_r)$ is an $\mathcal{O}$-regular sequence which generates $\mathfrak{m}$, then evidently $\mathrm{depth}(\mathcal{O}) \geq r$. Since $\dim(\mathcal{O}) \geq \mathrm{depth}(\mathcal{O})$, $\dim(\mathcal{O}) \geq r$. But $r \geq e$, and hence $\dim(\mathcal{O}) \geq e$. Since the reverse inequality is always true, (2) follows.

Suppose $\dim(\mathcal{O}) = e$. The homomorphism σ is always surjective, so it suffices to show that it is injective. Let K be its kernel, a homogeneous ideal of the symmetric algebra $S^{\cdot}$. If K is not zero, there is an $r > 0$ with a nonzero $f \in K$ of degree r. Then since $S^{\cdot}$ is isomorphic to a polynomial ring, it is an integral domain, and multipication by f defines an injective map from S^{i-r} to K_i. Then the dimension of K_i is at least the dimenison of S^{i-r}, and the dimension of the quotient G_i is at most the dimension $h(i)$ of S^i minus the dimension of S^{i-r}. Recall that for $i \geq 0$, the dimension of S^i is $p_{e-1}(i)$, so that $h(i) \leq p_{e-1}(i) - p_{e-1}(i-r)$, which is a polynomial of degree at most $e-2$. Thus $\ell_{\mathcal{O}}(i)$ is bounded by a polynomial of degree at most $e-1$, which contradicts the equality $e(\mathcal{O}) = \dim(\mathcal{O})$.

We prove that (3) implies (1) by induction on e. In fact we prove more: every sequence of generators for $\mathfrak{m}$ of length e is $\mathcal{O}$-regular, If this is zero, then $\mathfrak{m} = 0$ and so $\mathcal{O} = k$ and the statement is vacuous. For the induction step, assume that $e > 0$ and let $(x_1, \ldots, x_e)$ be a lift of a basis $(\overline{x}_1, \ldots, \overline{x}_e)$ of $\mathfrak{m}/\mathfrak{m}^2$. Then $\overline{x}_i = In(x_i)$ for all i. The assumption (3) implies that $\mathrm{Gr}_m(\mathcal{O})$ is an integral domain, and hence by the proposition, $\mathcal{O}$ is a domain also. Moreover, the proposition also implies that if I is the ideal of $\mathcal{O}$ generated by (x), then $In(I)$ is generated by $\overline{x}$, so that $\mathrm{Gr}_{\mathfrak{m}}(\mathcal{O}/(x_1) \cong \mathrm{Gr}(\mathcal{O})/(\overline{x}_1)$. It follows that the map

$$k[x_2, \ldots, x_e] \to \mathrm{Gr}_{\mathfrak{m}}(\mathcal{O}/(x_1)))$$

is again an isomorphsm. Then the induction hypothesis applies to tell us that the sequence $(x_2, \cdots, x_e)$ is $\mathcal{O}/(x_1)$ regular, and hence that $(x_1, \ldots, x_e)$ is $\mathcal{O}$-regular. □