Constructible, open, and closed sets

March 18, 2016

A topological space is *sober* if every irreducible closed set Z contains a unique point ζ such that the set $\{\zeta\}$ is dense in Z. (Such a point is called a "generic point of Z.") These points may seem to be fictional, but they are very useful in algebraic geometry. If A is a commutative ring, $\mathrm{Spec}(A)$ is always a sober topological space.

Let X be a sober space and S a subset of X. We consider the following condition.

(G) Every point s of S has a neighborhood U such that $U \cap \{s\}^-$ is contained in S.

Note that if S satisfies (G) and Z is a closed subset of X, the intersection $S \cap Z$ also satisfies (G) (as a subset of Z).

Theorem 1 *Let $\theta\colon A \to B$ be a homomorphism of commutative rings and let $S \subseteq Y := \mathrm{Spec}(A)$ denote the set of primes P which can be lifted to a prime of B.*

1. *The complement of S in Y satisfies (G).*
2. *If B is finitely generated as an A-algebra, then S satisfies (G).*

Proof: We give only a sketch. Suppose that $s \in Y \setminus S$. Then s corresponds to a prime ideal P of A and s is the generic point of the closed set Z of Y defined by P; in fact Z is homeomorphic to $\mathrm{Spec}(A/P)$. The set of prime ideals of B which lift P can be identified with $\mathrm{Spec}(B \otimes_A k(P))$, where $k(P)$ is the fraction field K of A/P. Since there are no such primes, the ring $B \otimes_A k(P)$ is the zero ring, *i.e.*, $B \otimes_K k(P) \cong (B/PB) \otimes_A K = 0$. This means that for every element b of B, there is some element $a \in A \setminus P$ such that $ab \in PB$. In particular, this applies with $b = 1_B$. Then 1_B belongs to PB_a, and it follows that $(B/PB)_a = 0$. But $\mathrm{Spec}(B/PB_a)$ is the set of prime ideals of Q which map to $D(a) \cap \mathrm{Spec}(A/PA)$, and so $D(a)$ is the desired neighborhood of s.

Suppose that $s \in S$. Then s corresponds to a prime ideal P such that $B \otimes_A k(P) \neq 0$, say $P = \theta^{-1}(Q)$, for $Q \in \mathrm{Spec}(B)$. Then we have an injection $A/P \to B/Q$, and the latter is finitely generated as an algebra over A/P. By Noether normalization, there is an injective and finite homomorphism $K[t_1, \ldots, t_n] \to B/Q$. Multiplying the images of the elements of t_i by suitable elements of A/P, we may assume that this homomorphism comes from a homomorphism $A/P[t_1, \ldots, t_n] \to B$. This homomorphism is necessarily injective. Choose a finite set of generators for B as an A-algebra. Each of these satisfies a monic polynomial with coefficients in $K[t_1, \ldots, t_n]$ and each of these can be viewed as a monic polynomial in $A/P_a[t_1, \ldots, t_n]$ for some $a \in A \setminus P$. Multiplying together the finite set of such elements a, we find one such a which works for all the generators. Then $A/P_a[t_1, \ldots, t_n] \to B_a$ is finite, hence every prime of $A/P_a[t_1, \ldots, t_n]$ lifts, and hence so do all primes of A/P_a. □

Theorem 2 *Let S be a subset of a noetherian and sober topological space X. Then S is constructible if and only if S and its complement satisfy (G).*

Proof: Let Y be a closed subset of X. Note that if S satisfies (G) in X, then $S \cap Y$ satisfies (G) as a subset of Y. Consider the family Σ of all closed subsets Y of X such that $Y \cap S$ is not constructible. The theorem asserts that X does not belong to Σ, so it is enough to prove that Σ is empty. Otherwise it has a minimal element, since X is noetherian. Thus we may assume without loss of generality that for every proper closed subset Y of X, $Y \cap S$ is constructible. We shall prove that, with this assumption, S is constructible, provided that it and its complement satisfy (G).

Let ξ be a generic point of X. We consider two cases. First suppose that $\xi \notin S$. Then since the complement of S satisfies (G), there is a neighborhood U of ξ such that $U \cap \{\xi\}^-$ does not meet S. Let $Z'' := \{\xi\}^- \setminus U$ and let Z' be the union of the irreducible components other than $\{\xi\}^-$. Then $\xi \notin Z'$, and $S \subseteq Z' \cup Z''$, a proper subset of X. It follows from the minimality that S is constructible. Next suppose that $\xi \in S$. Since S satisfies (G), there exists an open neighborhood U of ξ such that $U \cap \{\xi\}^- \subseteq S$. Let $Z'' := \{\xi\}^- \setminus U$. Then $Z'' \cup Z'$ is a proper closed subset of X, hence $S \cap (Z'' \cup Z')$ is constructible. But $Z'' \cup Z' \cup (U \cap \{\xi\}^-) = X$ and $(U \cap \{\xi\}^-) \subseteq S$, so $S = (S \cap (Z'' \cup Z') \cup (U \cap \overline{\xi})$ is the union of a constructible set and a locally closed set, hence is constructible. We omit the proof of the converse. □

Theorem 3 *Let S be a subset of a noetherian and sober topological space*

X. Assume that S satisfies (G) and that S is stable under generization. Then S is open.

Proof: Let Z be the closure of $X \backslash S$, with irreducible components $Z_1, \ldots, Z_n$, with respective generic points $\zeta_1, \ldots, \zeta_n$. We claim that no ζ_i belongs to S. Suppose for example that $\zeta_1 \in S$. Then by (G) there is an open neighborhood U of ζ_1 such that $U \cap Z_1 \subseteq S$. Let $Z'' := Z_1 \backslash U$ and let $Z' := Z_2 \cup \cdots Z_n$. Then $Z_1 \setminus Z'' \subseteq S$ and $X \setminus Z \subseteq S$. Hence

$$(X \setminus Z'') \cap (X \setminus Z') = (X \setminus Z) \cup (Z_1 \setminus Z'') \subset S$$

But then this set is an open neighborhood of ζ_1 contained in S, which contradicts the assumption that ζ_1 belongs to the closure of Z.

It follows that S is open. Indeed no point s of S can belong to Z; otherwise it would belong to some Z_i and hence some ζ_i would be a generization of s, hence would belong to S, contradicting our previous conclusion. □

Theorem 4 *Let $\theta: A \to B$ be a flat homomorphism of commutative rings. If θ is of finite type and A is noetherian, then the image of* $\mathrm{Spec}(\theta)$ *is open in* $\mathrm{Spec}(A)$.

Proof: In view of the above results, it is enough to show that the image of $\mathrm{Spec}(\theta)$ is closed under generization. Let P be a prime ideal of A which lifts to a prime ideal Q of B. Then θ induces a local homomorphism of local rings $\theta_Q: A_P \to B_Q$. The flatness of θ implies the flatness of θ_Q. But a flat local homomorphism of local rings is faithfully flat, and hence $\mathrm{Spec}(B_Q) \to \mathrm{Spec}(A_P)$ is surjective. Since the primes of $\mathrm{Spec}(A_P)$ correspond to the generizations of P, every such generization belongs to the image. □