Koszul complexes

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Let R be a commutative ring, let E be an R-module, and let q be a natural number. A *q-linear map* from E to an R-module F is a function $f\colon E^q \to F$ which is linear in each variable separately:

$$f(e_1, \dots, ae_i + a'e'_i, \dots, e_q) = af(e_1, \dots, e_i, \dots e_q) + a'f(e_1, \dots, e'_i, \dots, e_q).$$

There is a universal q-linear map

$$E^q \to T^qE : (e_1, \dots, e_q) \mapsto e_1 \otimes \cdots \otimes e_q.$$

If E is free with basis $(e_1, \dots e_n)$, then T^qE is also free, with basis the set of elements of the form $e_I = e_{I_1} \otimes e_{I_2} \otimes \cdots \otimes e_{I_q}$, where $1 \leq I_i \leq n$ for each n. Thus T^iE has rank n^q.

A q-linear map $f\colon E^q \to F$ is *alternating* if $f(e_1, \dots, e_q) = 0$ whenever $e_i = e_j$ for some $i < j$. This condition implies that f is in fact antisymmetric: interchanging any two elements in the sequence $(e_1, \dots, e_q)$ changes the sign of $f(e_1, \dots, e_q)$, and more generally applying a permutation to the sequence changes the sign of f by the sign of the permutation. There is a universal alternating q-linear map

$$E^q \to \Lambda^qE : (e_1, \dots, e_q) \mapsto e_1 \wedge \cdots \wedge e_q.$$

If $(e_1, \dots, e_n$ is a basis for E, then the set of elements $e_I := e_{I_1} \wedge \dots e_{I_q}$, with $I_1 < I_2 \cdots < I_q$ is a basis for Λ^qE. Thus Λ^qE is free of rank $\binom{n}{q}$.

There is a commutative diagram:

$$\begin{tikzcd}
E^p \times E^q \arrow[r, "\cong"] \arrow[d] & E^{p+q} \arrow[d] \\
\Lambda^pE \times \Lambda^qE \arrow[r] & \Lambda^{p+q}E
\end{tikzcd}$$

which gives the direct sum

$$\Lambda^{\cdot} E := \oplus_q \Lambda^q E$$

the structure of a skew-commutative algebra, with the multiplication law written as $(a, b) \mapsto a \wedge b$. Thus $a \wedge b = (-1)^{ij} b \wedge a$ if $a \in \Lambda^i E$ and $b \in \Lambda^j E$.

Now suppose that $\phi: E \to R$ is a homomorphism. For each q, the map

$$E^q \to \Lambda^{q-1} E : (e_1, \ldots e_q) \mapsto \sum_i \phi(e_i)(-1)^{i-1} e_1 \wedge \cdots \hat{e}_i \cdots \wedge e_q.$$

is q-linear and alternating, and hence factors through a homomorphism

$$\phi_q: \Lambda^q E \to \Lambda^{q-1} E,$$

(sometimes called *exterior multiplication*. One easily verifies that if $a \in \Lambda^i E$ and $b \in \Lambda^{q-j} E$, then

$$\phi_q(a \wedge b) = \phi_q(a) \wedge b + (-1)^i a \wedge \phi(b).$$

One can use this formula to prove easily by induction on q that $\phi_{q-1}\phi_q = 0$, so that we get a chain complex

$$K.(\phi) := \{(K_q, d_q) := (\Lambda^q E, \phi_q)\}.$$

Definition 0.1 *If E is an R-module and $\phi: E \to R$ a homomorphism, then the chain complex $K.(\phi)$ described above is the Kozul complex of ϕ. More generally, if M is an R-module, $K(\phi, M)$ is the tensor product $K.(\phi) \otimes M$.*

For example, if $E = R$ and ϕ is multiplication by x, then $K.(\phi, M)$ is the complex:

$$M \xrightarrow{x} M$$

placed in degrees 1 and 0. Thus $H_1(\phi, M)$ is the kernel of multiplication by x and $H_0(\phi, M)$ is the cokernel. In general, $K_0(\phi) = \Lambda^0 E = R$ and $\phi_1 = \phi: E \to R$. Thus $H_0(\phi) = Cok(\phi) = R/I$, where I is the image of $\phi: E \to R$.

An important example is the following. Let us start with an ideal I in R and choose a sequence $x. := (x_1, \ldots, x_n)$ of generators for I. Let $E := R^n$ and let $\phi: E \to R$ be the map determined by the sequence $x.$. Then the first two terms of $K.(\phi)$ form the start of a free resolution of R/I. The next term consists of the "trivial" relations among the generators—in general there may other relations as well, so the Koszul complex may not a resolution of R/I. We will see below that it is a resolution if the sequence $x.$ is a regular sequence. First we shall make some general remarks.

Proposition 0.2 *Let* $\phi: E \to R$ *be a homomorphism and let* M *be an* R*-module and let* $I := Im(\phi: E \to R)$.

1. $H_0(\phi, M)$ *is canonically isomorphic to* M/IM.
2. *For every* q, $H_q(\phi, M)$ *is annihilated by* I.
3. *The differentials of the complex* $K_q(\phi, M) \otimes R/I$ *are all zero.*

Proof: The first statement is clear. For the second, it is enough to show that for every $e \in E$, multiplication by $\phi(e)$ acts as zero on $H_q(\phi, M)$. In fact we shall show that $\phi(e)$ acting on $K.(\phi, M)$ is homotopic to zero. Note that multiplication by e defines a family of homomorphisms:

$$\rho_e^q: \Lambda^q E \to \Lambda^{q+1} E: \omega \mapsto e \wedge \omega$$

which we can view as a homotopy operator ρ_e on the complex $K_q(\phi)$. One computes immediately that $\phi\rho_e + \rho_e\phi$ is just multiplication by $\phi(e)$ on $K.(\phi, M)$, as claimed. To prove the third statement, observe from the definition that $\phi_q(e_1, \ldots, e_q) \in I\Lambda^{q-1}$, and hence ϕ_q induces the zero map when reduced modulo I. $\square$

We asserted above that the Kozul complex $K.(\phi)$ is built out of the trivial relations amount a set of generators. This suggests that it should be contained in any resolution of R/I. In fact this is the case in the sense of the following proposition.

Proposition 0.3 *Let* E *be a finitely generated projective* R*-module, let* $\phi: E \to R$ *be a homomorphism, and let* I *be an ideal of* R *containing the image of* ϕ. *Then the Kozsul complex* $K(\phi)$ *admits an augmentation* $K(\phi) \to R/I$, *and the boundary maps of* $K(\phi)$ *are zero modulo* I. *Suppose that* $F. \to R/I$ *is any projective resolution of* R/I *whose boundary maps are zero modulo* I, *and let* $f.: K(\phi) \to F.$ *be any homomorphism of complexes compatible with the augmentations. Suppose that* I/I^2 *is a flat* R/I*-module. Then for each* q, $f_q \otimes \mathrm{id}_{R/I}$ *is injective.*

Proof: We argue by induction on q. If $q = 0$ we have a commutative diagram:

$$\begin{tikzcd} R \arrow[r] \arrow[d, "f_0"'] & R/I \arrow[d, "\mathrm{id}"] \\ F_0 \arrow[r, "\epsilon"] & R/I \end{tikzcd}$$

Then $\epsilon \circ f_0$ becomes the identity modulo I, and it follows that $f_0 \otimes \mathrm{id}_{R/I}$ is injective.

If $q = 1$, we use the fact that the boundary maps d_1 of $K(\phi)$ and $F.$ are zero modulo I to get a commutative diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "f_1"'] & I \arrow[d, "f_0'"] \\ F_1 \arrow[r] & IF_0, \end{tikzcd}$$

where f_0' is the homomorphism induced by f_0. Since F_0 is projective, the natural map $I \otimes F_0 \to IF_0$ is an isomorphism, and this square can be rewritten as follows:

$$\begin{tikzcd} E \arrow[r] \arrow[d, "f_1"'] & I \otimes R \arrow[d, "\mathrm{id}_I \otimes f_0"] \\ F_1 \arrow[r] & I \otimes F_0, \end{tikzcd}$$

When reduced modulo I, this square becomes:

$$\begin{tikzcd} E/IE \arrow[r] \arrow[d, "f_1 \otimes \mathrm{id}_{R/I}"'] & I/I^2 \otimes R/I \arrow[d, "\mathrm{id}_{I/I^2} \otimes (f_0 \otimes \mathrm{id}_{R/I})"] \\ F_1/IF_1 \arrow[r] & I/I^2 \otimes F_0/IF_0 \end{tikzcd}$$

We have already observed that $f_0 \otimes \mathrm{id}_{R/I}$ is injective, and since I/I^2 is flat, the vertical arrow on the right is also injective. The top horizontal arrow is also injective by assumption, and it follows that $f_1 \otimes \mathrm{id}_{R/I}$ is also injective.

For the general inductive step, we first note that for each q, there is a homomorphism

$$\psi_q\colon \Lambda^q E \to E \otimes \Lambda^{q-1} E : e_1 \wedge \cdots e_q \mapsto \sum_i (-1)^{i-1} e_i \otimes e_1 \wedge \cdots \hat{e}_i \wedge e_q.$$

I claim that this map is a split monomorphism. I only give the proof when E is free. In this case we choose a basis $(e_1, \ldots, e_n)$ for E. Then a basis for Λ^q is given by the set of elements of the form $e_I := e_{I_1} \wedge \cdots e_{I_q}$, where I is a multiindex such that $I_1 < I_2 < \ldots < I_q$. A basis for $E \otimes \Lambda^{q-1}E$ is thus given by elements of the form $e_i \otimes e_J$ where J is a multiindex $J_1 < J_2 < J_{q-1}$. Then we can define a section of our map by sending $e_i \otimes e_J$ to $e_i \wedge e_J$ if $i < J_1$ and to zero otherwise.

Now suppose that $q \geq 2$, and note that the boundary map ϕ_q of K_q is the compositite of ψ_q and $\phi \otimes \mathrm{id}$. Thus we get a commutative diagram:

$$\begin{tikzcd}
K_q E \arrow[r, "\psi_q"] \arrow[d, "f_q"'] & E \otimes K_{q-1} \arrow[d, "\phi \otimes f_{q-1}"] \\
F_q \arrow[r, "d_q"] & R \otimes F_{q-1}
\end{tikzcd}$$

The map on the bottom factors by assumption through $IF_{q-1} \cong I \otimes F_{q-1}$, and ϕ factor through a map $\tilde{\phi}\colon E \to I$. This gives a diagram

$$\begin{tikzcd}
K_q E \arrow[r, "\psi_q"] \arrow[d, "f_q"'] & E \otimes K_{q-1} \arrow[d, "\tilde{\phi} \otimes f_{q-1}"] \\
F_q \arrow[r, "\tilde{d}_q"] & I \otimes F_{q-1},
\end{tikzcd}$$

which reduces to

$$\begin{tikzcd}
K_q E \otimes R/I \arrow[r, "\psi_q"] \arrow[d, "f_q"'] & E/IE \otimes K_{q-1}/IK_{q-1} \arrow[d, "(\tilde{\phi} \otimes \mathrm{id}_{R/I}) \otimes (f_{q-1} \otimes \mathrm{id}_{R/I})"] \\
F_q/IF_q \arrow[r, "\tilde{d}_q"] & I/I^2 \otimes F_{q-1}/IF_{q-1},
\end{tikzcd}$$

Since ψ_q and ϕ and f_{q-1} are injective mod I and the modules are flat modulo I, it follows that the composite arrow is injective mod I, and hence so is f_q. $\square$

Corollary 0.4 *Let R be a noetherian local ring with maximal ideal m and residue field k. Let $e := \dim m/m^2$. Then $Tor_q(k,k)$ has dimenison at least $\binom{e}{q}$ for every q. In particular $Tor_e(k,k) \neq 0$ and $hd(k) \geq e$.*

Let $E' \to E$ be a homomorphism of R-modules. Then a homomorphism $\phi: E \to R$ induces a homomorphism $\phi': E \to R$, and it is clear from our construction that one for every M one gets a homomorphism of Koszul complexes:

$$K.(\phi', M) \to K.(\phi, M)$$

(functorially in M). Let I' be the image of ϕ'. Then if E'' is the cokernel of $E' \to E$, one has a commutative diagram

$$\begin{tikzcd}
E' \arrow[r] \arrow[d] \arrow[dr, "\phi'"] & E \arrow[r] \arrow[d, "\phi"] & E'' \arrow[d] \\
I' \arrow[r] & R \arrow[r] & R/I'
\end{tikzcd}$$

Suppose now that we have a split exact sequence

$$0 \to E' \xrightarrow{f} E \xrightarrow{g} E'' \longrightarrow 0.$$

The cokernel of $\Lambda^q E' \to \Lambda^q E$ is in general not so easy to describe. When E'' is projective of rank one, however, it is not difficult. I claim that in this case, there is a natural isomorphism

$$E'' \otimes \Lambda^{q-1} E' \to \Lambda^q E / \Lambda^q E' : e' \otimes \omega' \mapsto s(e') \wedge f(\omega'),$$

where $s: E'' \to E$ is any splitting of g. (The inverse is induced by the map

$$\Lambda^q E \to E'' \otimes \Lambda^{q-1} E' : e_1 \wedge \cdots \wedge e_q \mapsto \sum_i g(e_i)(-1)^{i-1} t(e_1) \wedge \cdots \hat{e}_i \wedge \cdots t(e_q),$$

where $t: E \to E'$ is the splitting induced by s.

Lemma 0.5 *Let $0 \to E' \to E \to E'' \to 0$ be a short exact sequence of R-modules, where E'' is projective of rank one, let $\phi: E \to R$ be a homomorphism, inducing a homomorphism $\phi': E' \to R$.*

1. *There is an exact sequence of complexes*

$$0 \to K.(\phi') \to K.(\phi) \to K.(\phi', E'')[-1] \to 0$$

2. *Let s be any splitting of g, inducing then a term-by-term splitting of the exact sequence of complexes*

$$0 \to K.(\phi'') \to K.(\phi) \to K.(\phi', E'') \to 0.$$

Then the corresponding map of complexes

$$w: K.(\phi', E'')[-1] \longrightarrow K.(\phi')[-1]$$

is the identity tensored with $\phi'' := \phi \circ s: E'' \to R$.

3. *For any M, there is a corresponding long exact sequence of cohomology:*

$$\cdots H_q(\phi', M) \to H_q(\phi, M) \to H_{q-1}(\phi', M) \otimes E'' \xrightarrow{\phi''} H_{q-1}(\phi', M) \to \cdots$$

Note that $H_q(\phi', M)$ is annihilated by I', and hence is naturally an R/I'-module. This implies that the map ϕ'' acting on the cohomology above is independent of the choice of the splitting. Furthermore, the kernel and cokernel of this map can be interpreted as the Koszul homology of ϕ''. Thus we find exact sequences

$$0 \to H_0(\phi'', H_q(\phi', M)) \to H_q(\phi, M) \to H_1(\phi'', H_{q-1}(\phi', M)) \to 0 \quad (1)$$

Theorem 0.6 *Let $(x_1, \ldots, x_r)$ be a sequence of elements in R, let I be the ideal it generates, and let*

$$\phi: E := R^r \to R$$

be the map sending e_i to x_i. Finally, let M be an R-module.

1. *$H_0(\phi, M) \cong M \otimes R/I$*

2. *If $(x_1, \ldots, x_r)$ is M-regular, $H_i(\phi, M) = 0$ for $i > 0$.*

3. *If M is noetherian and R is local and all x_i belong to its maximal ideal, then the converse of (2) is true. In fact it is enough to assume that $H_1(\phi, M) = 0$.*

Proof: The first statement is clear. We prove the second statement by induction on r. If $r = 1$ the statement is clear from the explicit description of the Koszul complex. For the induction step, let E' be the submodule of E spanned by the first $r-1$ basis vectors. Then ϕ' corresponds to the sequence $(x_1, \ldots, x_{r-1})$ which is M-regular. It follows from the induction assumption that $H_q(\phi', M) = 0$ for $q > 0$. Then the sequence 1 above shows that $H_q(\phi, M) = 0$ for $q \geq 2$. Moreover, $H_0(\phi', M) \cong M/I'M$, where I' is generated by $(x_1, \ldots, x_{r-1})$, and by hypothesis, x_r acts injectively on this module, Thus $H_1(\phi'', H_0(\phi', M)) = 0$, and hence $H_1(\phi, M)$ also vanishes.

We also prove the converse by induction on r. If $r = 1$, $H_1(\phi, M)$ identifies with the kernel of multiplication by x_1, so its vanishing implies that x_1 is M-regular. For the induction step, we again use the exact sequence (1). The vanishing of $H_1(\phi, M)$ implies that $H_1(\phi, H_0(\phi', M)) = 0$ and hence that x_r acts injectively on $M/I'M$. But it also implies the vanishing of $H_0(\phi'', H_1(\phi', M)) = H_1(\phi', M) \otimes R/(x_r)$. Since $H_1(\phi', M)$ is finitely generated and x_r belongs the maximal ideal, Nakayama's lemma implies that $H_1(\phi', M) = 0$. Then the induction hypothesis implies that $(x_1, \ldots, x_{r-1})$ is M-regular. □

Theorem 0.7 *Let R be a noetherian local regular ring of dimension d and let k be its residue field. Then $pd(k) = d$, and for every finitely generated R-module M, $pd(M) \leq d$.*

Proof: Since R is regular, its maximal ideal is generated by d elements, which form a regular sequence. The corresponding Koszul complex is then exact, and gives a projective resolution of length d of k. This shows that $pd(k) \leq d$. On the other hand, all the boundary maps in the Koszul complex reduce to zero when tensored with k, so $Tor_i(k, k)$ is not zero for $i \leq d$. This shows that the projective dimension of k is exactly d. Moreover, for every M, $Tor_i(k, M) = 0$ for $i > d$ and hence $Tor_i(M, k) = 0$ for $i > d$. When M is finitely generated, this implies that $pd(M) \leq d$.

□