# Dimension of Local Rings

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For each natural number $n$, consider the polynomial

$$p_n(t) := \binom{t+n}{n} = \frac{(t+n)(t+n-1)\cdots(t+1)}{n!} \in \mathbf{Q}[t].$$

Note that

$$p_n(t) = \frac{t^n}{n!} + \cdots + 1.$$

It follows that the set of polynomials $\{p_0(t), \ldots, p_n(t)\}$ forms a basis for the space of polynomials in $\mathbf{Q}[t]$ of degree at most $n$. For any $f \in \mathbf{Q}[t]$, let

$$\Delta(f)(t) := f(t) - f(t-1)$$

Note that the degree of $\Delta f$ exactly one less than the degree of $f$. Furthermore,

$$\Delta p_n = p_{n-1}$$

Let $\mathbf{Q}^{\mathbf{N}}$ denote the ring of functions $\mathbf{N} \to \mathbf{Q}$. We define an equivalence relation on this set by saying that $f \sim g$ if $f(i) = g(i)$ for all $i$ sufficiently large. This is the quotient by the ideal of functions which are eventually zero, and so the set of equivalence classes is again a ring, which we denote by $\mathcal{A}$ The evident map from the set of polynomials into $\mathcal{A}$ injective.

If $f \in \mathbf{Q}^{\mathbf{N}}$, we can define $\Delta(f)$ by $\Delta(f)(i) = f(i) - f(i-1)$, and $\Delta$ induces a maps $\mathcal{A} \to \mathcal{A}$. Note that if $\Delta(f) = 0$, then $f$ is constant.

**Lemma 0.1** *If $f \in \mathbf{Q}^{\mathbf{N}}$, then $f$ is equivalent to an element of $\mathbf{Q}[t]$ if and only if $\Delta f$ is.*

*Proof:* Suppose $\Delta(f) \sim g \in \mathbf{Q}[t]$. Write $g = \sum a_n p_n$ with $a_n \in \mathbf{Q}$. Then $\Delta(f) = \sum a_n \Delta p_{n+1}$. Let $h := \sum a_n p_{n+1}$, so that $\Delta(f) = \Delta(g)$. Hence $f - h$ is eventually constant, and it follows that $f \sim h + c$ for some $c$. $\square$

Let $A$ be a noetherian local ring with maximal ideal $\mathfrak{m}$ and let $M$ be a finitely generated $A$-module. An ideal $I$ of $A$ is said to be an *ideal of definition* of $A$ if $I \subseteq \mathfrak{m}$ and $I$ contains some power of $\mathfrak{m}$. Then $A/I$ has finite length, and hence so does $A/I^i$ for every $i$. Recall that a filtration $F$ on $M$ is said to be $I$-*stable* if $IF^iM \subseteq F^{i+1}M$ for all $i$, with equality for all $i$ sufficiently large. Given such a filtration, $M/F^{i+1}M$ has finite length for all $i$, and we set

$$\ell_{M,F}(i) := \ell(M/F^{i+1}M).$$

**Proposition 0.2** *With the notation above, there exists a polynomial $p_{M,F} \in \mathbf{Q}[t]$ of degree less than or equal to the number of generators of $I$, such that*

$$\ell_{M,F}(i) = p_{M,F}(i) \quad \textit{for all } i >> 0$$

*Proof:* Consider the graded ring $G_I(A) := \sum I^i/I^{i+1}$. This ring is generated over $G_0 = A/I$ by $I/I^2$, and can be regarded as a quotient of the polynomial ring $G := G_0[t_1, \ldots, t_n]$, where $n$ is the number of generators of $I$. Since $F$ is $I$-stable, the graded $G_I(A)$-module $G_I(M) := \sum F^iM/F^{i+1}M$ is finitely generated over $G_I(A)$ and hence also over $G$. For each $i$ we have an exact sequence

$$0 \to G_i(M) \to M/F^{i+1}M \to M/F^iM \to 0$$

Hence for all $i$,

$$\Delta\ell_{M,F}(i) = \ell(G_i(M)).$$

By Lemma 0.1, it suffices to show that there is a $q \in A_0$ of degree less than $n$ such that $q(i) = \ell(G_i(M))$ for all $i$ sufficiently large. Thus it suffices to prove the following result. □

**Lemma 0.3** *Let $R$ be an Artinian local ring, let $N$ be a finitely generated graded module over the graded ring $R[t_1, \ldots, t_n]$, and let $h_N(i) := \ell(N_i)$. Then there is a unique polynomial $p_N \in \mathbf{Q}[t]$ such that $p_n(i) = h_N(i)$ for all $i$ sufficiently large. The degree of $p_N$ is at most $n-1$.*

*Proof:* By induction on $n$. If $n = 0$, $N_i = 0$ for $i$ sufficiently large, so $h_n \sim 0$. If $n > 0$, let $t := t_n$ and consider the exact sequence of graded modules

$$0 \to K \to N \xrightarrow{t} N \to \overline{N} \to 0$$

Here multiplication by $t$ increases degrees, and by additivity of lengths we have that

$$h_K(i-1) + h_N(i) = h_N(i-1) + h_{\overline{N}},$$

*i.e.*, that

$$\Delta h_N = h_{\overline{N}} - h_{K(-1)}$$

Since $K$ and $\overline{N}$ are killed by $t$, the induction hypothesis applies to them, and it follows that $\Delta h_N$ is given by a polynomial of degree at most $n-2$. Hence $h_N$ is given by a polynomial of degree at most $n-1$. □

**Lemma 0.4** *The degree and leading term of $\ell_{M,F}$ are independent of the $I$-stable filtration $F$, and the degree is independent of the ideal $I$.*

*Proof:* For the first statement, recall that $I^iM \subseteq F^iM$ for all $i$ and that for some $r$, $F^{i+r}M \subseteq I^iM$ for all $i \geq 0$. We deduce

$$\ell_{M,I}(i) \geq \ell_{M,F)(i)} \quad \text{and} \quad \ell_{M,F(i+r)} \geq \ell_{M,I}(i)$$

Hence for large enough $i$,

$$p_{M,I}(i) \geq p_{M,F}(i) \quad \text{and} \quad p_{M,F}(i+r) \geq p_{M,I}(i)$$

Expanding out the polynomials, we find the desired conclusion

For the second statement, observe first that $\ell_{M,I}(i) \geq \ell_{M,\mathfrak{m}}(i)$ for all $i$. This implies that the degree of $p_{M,I}$ is at least the degree of $p_{M,\mathfrak{m}}$. On the other hand, if $m^r \subseteq I$, then $\mathfrak{m}^{ir} \subseteq I^i$ for all $i$, and it follows that $\ell_{M,I}(i) \leq \ell_{M,\mathfrak{m}}(ir)$ for all $i$. Hence for $i$ large,

$$p_{M,I}(i) \leq p_{M,\mathfrak{m}}(ri)$$

Thus the degree of $p_{M,\mathfrak{m}}(rt)$ is at least the degree of $p_{M,I}(t)$. □

For a finitely generated $A$-module $M$, we denote by $d(M)$ the degree of the polynomial $p_{M,F}$ for any $I$-stable filtration $F$ as above and any ideal of definition $I$ of $A$.

**Corollary 0.5** *If $M' \subset M$, then $d(M') \leq d(M)$. If $x \in \mathfrak{m}$ is a nonzero divisor of $M$, then $d(M/xM) < d(M)$.*

*Proof:* Let $F$ be an $I$-stable filtration on $M$ and endow $M$; with the induced filtration. By Artin-Rees, this filtration is again $I$-stable, so so $d(M')$ is the degree of $p_{M',F}$. Since $\ell_{M'}(i) \leq \ell_M(i)$, $d(M') \leq d(M)$. Now let $M' = xM$. Since $x$ is a nonzero divisor, $M' \cong M$, and the filtration $F$ filtration on $M'$ is $\mathfrak{m}$-stable. Hence the degree and leading coefficient of $p_{N,F}$ agree with those of $p_{M,\mathfrak{m}}$. The strict exact sequence

$$0 \to (M', F) \to (M, F) \to (\overline{M}, F) \to 0$$

then shows that $p_{M,F} - p_{M',F} = p_{\overline{M},F}$, and hence that the degree of the latter is strictly less than the degree of $p_{M,F}$. □

**Definition 0.6** *If $M$ is a nonzero finitely generated $A$-module, then*

- $\dim(M)$ *is the supremum of the set of all $k$ such that there exists a chain of prime ideals of length $k$ contained in the support of $M$.*
- $d(M)$ *is the degree of the polynomial $p_{M,\mathfrak{m}}$ defined above.*
- $s(M)$ *is the minimim number of element of $\mathfrak{m}$ needed to generate an ideal $I$ such that $M/IM$ has finite length.*

**Theorem 0.7** *The above numbers are all equal.*

*Proof:* Step 1: $\dim(M) \leq d(M)$.

By induction on $d(M)$. If $d(M) = 0$, then $M$ has finite length and so its support is just $\{\mathfrak{m}\}$, and $\dim(M) = 0$. For the induction step, let $P_0 \subseteq P_1 \subseteq \cdots P_k$ be a chain of prime ideals in *supp*$(M)$. We claim that $k \leq \dim(M)$. Without loss of generality we may assume that $P_0$ is a minimal prime of *supp*$(M)$ and hence is associated to $M$. Then there exists an embedding $A/P_0 \subseteq M$ and hence $d(A/P_0) \leq d(M)$. Thus it suffices to prove the result with $M = A/P_0$, which we henceforth assume. If $P_0 = \mathfrak{m}$ there is nothing to prove, and othewise we can choose $x \in P_1 \setminus P_0$. Then if $\overline{M} := M/xM$, it follows that $d(\overline{M}) < d(M)$ and hence by the induction assumption, $\dim(\overline{M}) \leq d(\overline{M})$. But $P_1 \in supp(\overline{M})$ so $\dim(\overline{M}) \geq k-1$. Thus $k - 1 \leq d(\overline{M}) \leq d(M) - 1$, □.

Step 2: $d(M) \leq s(M)$.

Let $J$ be the annihilator of $M$. Then $M$ is a faithful $A/J$-module, and we may without loss of generality replace $A$ by $A/J$. Choose a sequence

$(x_1, \ldots, x_s)$ of elements in the maximal ideal of $A$ such that $M/IM$ is Artinian, where $I := (x_1, \ldots, x_s)$. Then $m^n M \subseteq IM$ for some $n$. We claim that $I$ is an ideal of definition of $A$, equivalently, that $m$ is the only prime ideal of $A$ containing $I$. We need the following useful lemma.

**Lemma 0.8** *Let $E$ be a finitely generated module over a commutative ring $R$ and let $I$ be an ideal of $R$. Then*

$$supp(E/IE) = supp(E) \cap supp(R/I)$$

*Proof:* Let $P$ be a prime ideal of $R$. Since $E/IE \cong E \otimes A/I$, it is clear that $(E/I)_P$ vanishes if $E_P$ or $(A/I)_P$ vanishes, so $supp(E/IE) \subseteq (E) \cap (R/I)$. Suppose on the other hand that $I \subseteq P$ and that $E_P \neq 0$. By Nakayama, $E(P) := E_P/PE_P \cong (E/PE)_P$ is not zero, and since $I \subseteq P$, $E/PE \cong (E/I)/P(E/I)$, and hence $E/IE(P) \cong (E/PE)P \neq 0$. □

In our case, $M$ is faithful, so $supp(M) = \operatorname{Spec}(A)$, and since $supp(M/IM) = \{m\}$, it follows that $supp(A/I) = \{m\}$. Then $d(M)$ is the degree of $p_{M,I}$, which is at most $s$, by Proposition 0.2.

Step 3: $s(M) \leq \dim(M)$. By induction on $\dim(M)$. If this is zero, the support of $M$ is $\{\mathfrak{m}\}$, hence $M$ is annihilated by a power of $\mathfrak{m}$, hence $M$ has finite length and we can take the empty sequence for $(x_1, \ldots, x_s)$. Thus $s = 0$. If $\dim(M)$ is positive, then the support of $M$ contains a finite number of minimal primes (finite since all are associated to $M$) none of which is $\mathfrak{m}$ and it follows by prime avoidance that there exists an $x \in \mathfrak{m}$ which does not belong to any such minimal prime. Then the support of $M/xM$ does not contain any of the minimal primes of $supp(M)$ and hence $\dim(M/xM) < \dim(M)$. By the induction assumption, $s(M/xM) \leq \dim(M/xM) \leq \dim(M)-1$, so there is a sequence $(x_1, \ldots x_{s-1})$ with $s \leq \dim(M)-1$ such that $(M/xM/(x_1, \ldots, x_{s-1})$ has finite length. Then $M/(x, x_1, \ldots, x_{s-1})M$ has finite length.

□