Colimits and Localization

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For some reason, calculating colimits, say in the category of sets, seems to be more difficult that calculating limits: forming colimits require taking the "quotient" by an equivalence relation which can be difficult to make explicit. The calculations are much easier when the following conditions are fulfilled.

Definition A category I is said to be *filtering* if it satisfies the following conditions:

1. It is not empty.
2. For any two objects i and j, there exists arrows a and b such that $s(a) = i, s(b) = j$, and $t(a) = t(b)$:
3. For any two arrows a and b with the same source and target, there exists an arrow c such that $ca = cb$.

For example, the category $\mathbf{N}$ of natural numbers is filtering. Its opposite, is also filtering but it in a trivial way, in that it has final object. (Observe that, in general, if I has a final object o, then for every i there is a unique arrow $a_i\colon i \to 0$, and if $C.$ is an I-system, $\mathrm{colim}(C.)$ is just C_0, and $\{C_{a_i} : i \in I\}$ is the universal family.)

Colimits over filtering categories are sometimes called *direct limits*. The following result gives an explicit description of colimits over filtering categories in the category of sets.

Theorem Let I be a filterting category and let $S.$ be an I-system of sets. Let S_* denote the disjoint union of all the sets S_i and let $E \subseteq S_* \times S_*$ be the set of pairs $(s_i, s_j) \in S_* \times S_*$ such that there exist arrows a and b in I such that $Source(a) = i$, $Source(b) = j$, $Target(a) = Target(b)$, and $S_a(s_i) = S_b(s_j)$. Then E is an equivalence relation, the quotient S_*/E is a colimit of $S.$, and the evident family of maps $q_i\colon S_i \to S_*/E$ is a universal compatible family

Corollary If I is filtering, forming the colimit over I commutes with the forget functor from the category of abelian groups to the category of sets.

Example. Let M be a monoid and let S be an M-set. The transporter category of S is the category whose objects are the elements of S, and for $s, s' \in S$, the arrows from s to s' are the elements m of M such that $ms = s'$. Then multiplication in M defines a composition law to make this collection of objects and arrows into a category. Let us check that if M is commutative and if S is M, acting on itself, then the transporter category is filtering. It is not empty because M contains a unit element. Suppose that s and t are elements of S. Then $st = ts$, and t maps s to $ts = st$ and s maps t to $st = ts$, so (2) is satisfied. Suppose next that a and b are arrows from s to t, so that $as = t$ and $bs = t$. Let $u := st$. Then s is an arrow from t to u and $sa = sb$, so (3) is also satisfied. It is however not true that the transporter category of every M-set is filtering. A not so obvious theorem asserts that the transporter category of an M-set S is filtering if and only if S is a direct limit (filtered colimit) of free M-sets.

Localization Now let $\mathcal{C}$ be a category, and suppose that all filtered colimits exist in $\mathcal{C}$. Let E be an object of $\mathcal{C}$ and let S be a commutative monoid acting by endomorphisms of E. For $s \in S$ we denote the corresponding endomorphism of E by $\mu_E(s)$. For example, $\mathcal{C}$ might be the category of modules over a ring R, S might be a submonoid of the multiplicative monoid of R and E an R-module. Let I be the transporter category of S, viewed as an S-set acting on itself, and define an I-diagram $E.$ in $\mathcal{C}$ by sending every $i \in I$ to E and every arrow a to $\mu_E(a)\colon E_i = E \to E_{ai} = E$. Let $\{q_i\colon E_i \to L\}$ be the colimit, *i.e.*, the universal family of maps satifying the compatibility condition

$$q_i = q_{ti} \circ \mu_E(t) \tag{1}$$

for all $i \in I$ and all $t \in S$. The commutativity of S implies that, for every $s \in S$, $\mu_E(s)$ defines a map $E_i \to E_i$ which is compatible with all the maps q_i. By the universal property of L, we find a unique map $\mu_L(s)\colon L \to L$ such that $\mu_L(s) \circ q_i = q_i \circ \mu_{E_i}(s)$ for every i.

Theorem The object L above has the following properties.

1. For every $s \in S$, the arrow $\mu_L(s)\colon L \to L$ is an isomorphism.

2. The map $q_0\colon E \to L$ is compatible with the actions of S.

3. If $\alpha\colon E \to F$ is another arrow in $\mathcal{C}$, with S acting as isomorphisms on F, then there is a unique arrow $\theta\colon L \to F$ such that $\theta \circ q_0 = \alpha$.

Proof: To construct an inverse to $\mu_L(s)$, we use the following tricky argument. For each $i \in I$, recall that $E_i = E = E_{is}$, and so q_{is} can also be viewed as a map $\tilde{q}_i\colon E_i \to L$. Then $\{\tilde{q}_i\colon E_i \to L\}$ is another family of compatible maps, which then induces a map $\tilde{s}\colon L \to L$, uniquely determined by the fact that $\tilde{s} \circ q_i = \tilde{q}_i$ for all i. We claim that $\tilde{s} \circ \mu_L(s) = \mu_L(s) \circ \tilde{s} = \mathrm{id}_L$. To check this, it is enough to see that the equalities hold after composing both sides with q_i. We compute:

$$\begin{aligned}
\mu_L(s) \circ \tilde{s} \circ q_i &= \mu_L(s) \circ \tilde{q}_i \\
&= \mu_L(s) \circ q_{is} \\
&= q_{is} \circ \mu_E(s) \\
&= q_i,
\end{aligned}$$

using equation (1) and the commutatiivty of S. Similarly;

$$\begin{aligned}
\tilde{s} \circ \mu_L(s) \circ q_i &= \tilde{s} \circ q_i \circ \mu_E(s) \\
&= q_{is} \circ \mu_E(s) \\
&= q_i,
\end{aligned}$$

Statement (2) is built in the construction. For (3), suppose that α is given. Construct the I-diagram $F.$ in the same way that we did for E. By hypothesis, all the arrows $F_i \to F_{is}$ are isomorphisms. Note that the identity element 1 of S defines an initial object 0 of I: for every $i \in I$, there is a unique arrow $a_i\colon 0 \to i$ (namely i). Let $q'_i := F_{a_i}^{-1}\colon F_i \to F_0$. Then this family is compatible, and it follows that the composition $E_i \to F_i \to F_0 = F$ also forms a compatible family. This family induces a morphism $L \to F$, and we leave the rest of the verifications to reader. $\square$

Let us return to the more down-to-earth case of R-modules. We should compare the categorical construction given here with the usual construction of a localization of an R-module E by a multiplicative subset S of R. Typically this is done by taking the quotient of the product $E \times S$ by the equivalence relation given by :

$$(e, s) \sim (e', s') \text{ iff there exist } s'' \in S : s''s'e = s''se' \tag{2}$$

(We think of the equivalence class of (e, s) as a ratio e/s.) Now the above construction says to take the colimit over the I-diagram $E.$. Let us apply the construction in the theorem, which says to form the disjoint union of the sets E_i and then divide by a certain equivalence relation. Since the objects of I are the same as the elements of S and since $E_i = E$ for every i, this disjoint union is exact the same as $E \times S$. What is the equivalence relation? It says

$$(e, s) \sim (e', s') \text{ iff there exist } t, t' \in S : ts = t's' \text{ and } te = t'e' \tag{3}$$

It is perhaps not obvious that the equivalence relations (2) and (3) are the same. Suppose that (3) holds. Then

$$ts'e = s'te = s't'e' = t's'e' = tse'$$

Thus if we take $s'' := t$, we see that (2) holds. Suppose that (2) holds. Then take $t' := s''s$ and $t := s''s'$. Then

$$ts = s''s's = s''ss' = t's' \text{ and } te = s''s'e = s''se' = t'e'$$