Lazard's theorem

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Let R be a commutative ring and let E and F be R-modules. Recall that there is a natural homomorphism

$$\mathrm{Hom}(E, R) \otimes F \to \mathrm{Hom}(E, F)$$

sending a pair (ϕ, f) to the homomorphism h given by $h(e) := \phi(e)f$. This homomorphism is an isomorphism if E is finitely generated and free, as is easy to see. For any E, let $E^\vee := \mathrm{Hom}(E, R)$. Then there is a natural transformation $E \to (E^\vee)^\vee$, which is also an isomorphism if E is finitely generated and free. Note that the functor $E \mapsto E^\vee$ is not exact in general.

Our aim is to prove and exploit the fact that an R-module M is flat if and only if every relation in E is trivial. First let us explain what this means. Let $(x_1, \ldots, x_m)$ be a sequence of elements in M. By a relation among the elements of this sequence we mean a sequence $(a_1, \ldots, a_m)$ in R such that $\sum a_j x_j = 0$. Such an equation is trivially true if there exist a sequence $(x'_1, \ldots, x'_n)$ in M and an $m \times n$ matrix $(b_{i,j})$ such that $x_j = \sum_i b_{i,j} x'_i$ for all j and $\sum_j b_{ij} a_j = 0$ for all i.

Let us try to restate this in an index-free way. Recall that to give a sequence of m elements in M is equivalent to giving a homomorphism $\phi \colon R^m \to M$. A relation among these is then a homomorphism $a \colon R \to R^m$ such that $\phi \circ a = 0$. The relation is trivial if there exist a homomorphism $b \colon R^m \to R^n$ and a homomorphism $\phi' \colon R^n \to M$ such that $b \circ a = 0$ and $\phi = \phi' \circ b$.

Theorem 1 *Let M be an R-module. Consider the the following conditions:*

1. *The R-module M is flat.*

2. *For every ideal I of R, the natural map $I \otimes M \to E$ is injective.*

3. *For every homomorphism of finitely generated free R-modules* $a\colon E_1 \to E_2$ *and every homomorphism* $\phi\colon E_2 \to M$ *such that* $\phi \circ a = 0$*, there exist a homomorphism of finitely generated free R-modules* $b\colon E_2 \to E_3$ *and a homomorphism* $\phi'\colon E_3 \to M$ *such that* $b \circ a = 0$ *and* $\phi = \phi' \circ b$.

4. *The same as above, but only when* E_1 *is free of rank one.*

5. M *is a direct limit of finitely generated free R-modules.*

6. *Every homomorphism* $N \to M$*, with* N *finitely presented, factors through a homomorphism* $F \to M$*, where* F *is free and finitely generated.*

Proof: Let us note that (1) easily implies (2), although we will see this implication again later in the course of the argument. It is also immediate to check that the slightly cumbersome statement (3) is equivalent to (6).

Proof that (1) implies (3):

Consider the dual map $a^\vee\colon E_2^\vee \to E_1^\vee$. Let K be its kernel, so that we have an exact sequence

$$0 \to K \to E_2^\vee \to E_1^\vee$$

Since M is flat, the resulting sequence

$$0 \to K \otimes M \to E_2^\vee \otimes M \to E_1^\vee \otimes M$$

is exact. Thus the element y of $E_2^\vee \otimes M \cong \mathrm{Hom}(E_2, M)$ corresponding to ϕ lies in $K \otimes M$ and can be written as a finite sum:

$$\phi = \sum_{i=1}^{n} k_i \otimes x_i,$$

with $k_i \in K \subseteq E_2^\vee$ and $x_i \in M$. Let $c\colon R^n \to E_2^\vee$ be the map given by the sequence $(k_1, \cdots, k_n)$. Thus, if $\delta_1, \cdots, \delta_n$ is the standard basis for R^n, c is the unique map such that $c(\delta_i) = k_i$ for all i. Then $a^\vee \circ c = 0$, since each $k_i \in K$. Let $E_3 := (R^n)^\vee$. Then $E_3^\vee = R^n$ and $b := c^\vee$ is a map $E_2 \to E_3$ and $b \circ a = c^\vee \circ (a^\vee)^\vee = (a^\vee \circ c)^\vee = 0$. We now have a commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & E_3^\vee \otimes M & \xrightarrow{c \otimes \mathrm{id}_M} & E_2^\vee \otimes M & \xrightarrow{a^\vee \otimes \mathrm{id}_M} & E_1^\vee \otimes M \\
 & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 \longrightarrow & \mathrm{Hom}(E_3, M) & \xrightarrow{\circ b} & \mathrm{Hom}(E_2, M) & \xrightarrow{\circ a} & \mathrm{Hom}(E_1, M)
\end{array}$$

Consider the element $x := \sum \delta_i \otimes x_i \in R^n \otimes M = E_3^\vee \otimes M$. We have $(c \otimes \mathrm{id}_M)(x) = \sum c(\delta_i) \otimes x_i = \sum k_i \otimes x_i = y$. Thus x corresponds to an element ϕ' of $\mathrm{Hom}(E_3, M)$ such that $\phi' \circ b = \phi$, as desired.

Proof that (2) implies (4): We argue in the same way as above; now E_1 has rank one, and we identify it and its dual with R. Let I be the image of the map $a^\vee \colon E_2^\vee \to R$. Then the exact sequence $0 \to K \to E_2^\vee \to E_1^\vee$ gives an exact sequence $0 \to K \to E_2^\vee \to I \to 0$, and hence an exact sequence

$$K \otimes M \to E_2^\vee \otimes M \to I \otimes M \to 0$$

By hypothesis, the map $I \otimes M \to M \cong E_1 \otimes M$ is injective, so we still have

$$K \otimes M \to E_2^\vee \otimes M \to E_1^\vee \otimes M$$

exact, and the argument proceeds as before.

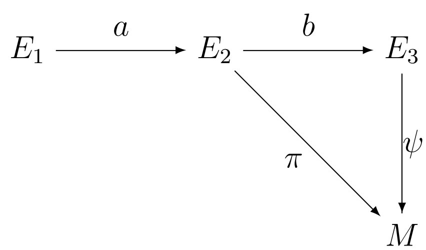
Statement (3) implies (4) trivially. Statement (4) gives (3) when the rank of E_1 is one, and one easily proves the general case by induction on this rank. Statement (4) implies (5) by the argument below in theorem (3). Statement (5) implies (1), because a direct limit of flat modules is flat. □

Corollary 2 *If M is finitely presented and flat, it is projective.*

Proof: If M is finitely presented, there is an exact sequence:

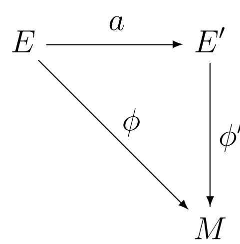
$$E_1 \xrightarrow{a} E_2 \xrightarrow{\pi} M \to 0.$$

If it is flat, then by the previous result there exist a free finitely generated R-module E_3 and a diagram as shown:



Let E' be the image of b. Then the map $E' \to M$ is surjective and injective, hence an isomorphism, and we see that M is a direct factor of E_3. □

Let M be an R-module, and consider the category I whose objects are pairs (E, ϕ), where E is a finitely generated free R-module and ϕ is a homomorphism $E \to M$, and where a morphisms $a\colon (E, \phi) \to (E', \phi')$ is a homomorphism such that the diagram below commutes.



For each $i := (E, \phi)$, let $E_i := E$ and for each $a\colon i \to j$, E_a is a. Thus $E.$ is an I-diagram in the category of R-modules. For each i, let $q_i\colon E_i \to M$ be ϕ; by construction, this family of homomorphisms is compatible with the transition maps in I, and hence defines a morphism $h\colon \operatorname{colim} E. \to M$.

Theorem 3 *The homomorphism h above is always an isomorphism. Thus every module M is a colimit of finitely generated free modules. If M is flat, the category I is filtering, so M is a direct limit of finitely generated free modules.*

Proof: We leave the first part of the theorem as an exercise.

Now suppose that M is flat. We check that if a and a' are arrows $i \to j$, then there is an arrow $b\colon j \to k$ such that $b \circ a = b \circ a'$. Replacing a by $a - a'$, we reduce to the case in which $a' = 0$. Thus we have (E_i, ϕ_i) and (E_j, ϕ_j) and a homomorphism $a\colon E_i \to E_j$ such that $\phi_j \circ a = \phi_j \circ a' = 0$. By the lemma, there exists a finitely generated free E_k and homomorphisms $b\colon E_j \to E_k$ and $\phi_k\colon E_k \to M$ such that $b \circ a = 0$ and $\phi_k \circ b = \phi_j$. $\square$