Math 250B Midterm:

March 1, 2016

1. (a) Let E be an R-module. Prove that the functor $T_E\colon M \to M \otimes E$ commutes with all colimits.
   Solution: This is because T_E has a right adjoint, h^E, taking N to $\mathrm{Hom}(E, N)$. Specifically, $\mathrm{Hom}(M \otimes E, N) \cong \mathrm{Hom}(M, \mathrm{Hom}(E, N))$.

   (b) Prove that a direct limit (filtering colimit) of flat modules is flat.
   Solution: Suppose that $E.$ is a direct system of flat R-modules and that $M' \to M$ is an injection of R-modules. Let $E := \varinjlim E.$. Then each $E_i \otimes M' \to E_i \otimes M$ is injective, since each E_i is flat. Since the direct limit of injections is injective and since tensor products commute with direct limits, we find that the map $E \otimes M' \to E \otimes M$ is injective, and hence that E is flat.

   (c) Show that a more general colimit of flat modules need not be flat. (Give a counterexample.)
   Solution: Let R be the ring of polynomials in one variable x over a field k. Then R is a free module over itself, hence flat. The coequalizer of 0 and multiplication by x on R is the quotient $R/(x)$, which is not flat, because $(x)/(x^2) \to R/(x)$ is not injective.

2. Let R be a commutative ring with identity, let $\mathcal{M}_R$ be the category of R-modules, and let F be the forgetful functor from the category of R-modules to the category of sets. Find a bijection from R to the set of natural transformations $F \to F$. (Hint: use Yoneda.)
   Solution: The functor F is represented by R itself. Then Yoneda tells us that the set of natural transformations from F to R is the same as the set of homomorphisms $R \to R$, which is just R.

3. Let R be a ring and let e be an element of R such that $e^2 = e$. Prove that the set $D(e)$ of all prime ideals P of R which do not contain e

is closed in the Zariski topology of R. Conclude (and explain) that if $e \neq 0$ and $e \neq 1$, then $\text{Spec}(R)$ is not connected.

Solution: Let $e' := 1 - e$. Then $ee' = 0$ and $e + e' = 1$. Let P be a prime ideal of R. The first of these equations implies that either e or e' belongs to P and the second that only of them does. Thus $\text{Spec}(R)$ is the disjoint union of the two sets $D(e)$ and $D(e')$. Since both of these are open, they are also closed. If $e \neq 1$, then $e' \neq 0$, and since $ee' = 0$, it follows that e is not a unit, and hence is contained some prime ideal. Thus $D(e')$ is not empty, and the same applies to $D(e)$. We have proved that $\text{Spec}(R)$ is the disjoint union of two nonempty open sets, and hence is disconnected.

4. Suppose that R is a local ring and that R/I is a flat R-module. Prove that either $I = 0$ or $I = R$. Note: Partial credit if you do this assuming that I is finitely generated.

   Solution: Let J be a finitely generated ideal contained in I. By the flatness assumption, the map $J/IJ \to R/I$ is injective. Since it is also the zero map, it follows that $J = IJ$. If $I \neq R$, it is contained in the maximal ideal of R, and it follows from Nakayama's lemma that $J = 0$. Since this is true for every finitely generated subideal of I, necessarily $I = 0$.